Week # 03, 04, 05

→ **Matrices**
→ **Operations on Matrices(Addition, subtraction, multiplication)**
→ **Transpose of Matrices**
→ **Invertible of Matrix**
→ **Determinants**
→ **Minors and Cofactors**

- **Matrix:-**

A matrix is a collection of numbers or functions arranged into rows and columns. Matrices are denoted by capital letters $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \; \ldots\ldots \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}.$ The numbers or functions are called elements of a matrix .The elements of matrix are denoted by small letters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \ldots\ldots \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}.$

Examples:-

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$ → Elements of Matrix

- **Rows And Columns:-**

The horizontal and vertical lines in a matrix are respectively called rows and columns of the matrix.

Examples:-

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$ → Horizontal Rows

↓ Vertical Columns

- **Order of a Matrix:-**

The size or dimensions of matrix is called order of matrix .Order of matrix is based on the numbers of rows and columns .It can be written as:

$$\boxed{\boldsymbol{r} \times \boldsymbol{c}}$$

r: Numbers of Rows
c: Numbers of Columns

Examples:-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \; 2 \times 3$$

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \; 3 \times 3$$

- **General Forms of Matrix:-**

$$X = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

- **Equality of Matrix:-**

The two matrices will be equal if they must have

a) The same dimension (same number or rows and columns) or same orer.

b) Corresponding elements must be equal.

Examples:-

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$a = w \ , \quad b = x$$
$$c = y \ , \quad d = z$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2-1 & 4-2 \\ 3+0 & 2+2 \end{bmatrix} \longrightarrow \text{Equal Matrices}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2-1 & 4-2 \\ 3+0 & 2+2 \end{bmatrix} \longrightarrow \text{Not Equal Matrices}$$

- **Questions:-**

Find $\boldsymbol{x}$ and $\boldsymbol{y}$ if **A** and **B** are equal matrices.

1. $A = \begin{bmatrix} 2x+1 & 3 & 4 \\ 0 & 3y+7 & 10 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 3 & 4 \\ 0 & 15 & 10 \end{bmatrix}$

$$A = B$$

$$\begin{bmatrix} 2x+1 & 3 & 4 \\ 0 & 3y+7 & 10 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 4 \\ 0 & 15 & 10 \end{bmatrix}$$

Compare $2x + 1 = 7$

$$2x + 1 = 7$$
$$2x = 7 - 1$$
$$2x = 6$$

Divide 2 on both sides

$$\frac{\cancel{2}x}{\cancel{2}} = \frac{\cancel{6}^{3}}{\cancel{2}^{1}}$$

$$\boxed{x = 3}$$

$$3y + 7 = 15$$
$$3y = 15 - 7$$
$$3y = 8$$
$$\boxed{y = \frac{8}{3}}$$

2. If $\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 2 \end{bmatrix}$

Find $a, \ b, \ c, \ d$

$Sol:-$

$a + b = 4 \longrightarrow$ (i)

$c + d = 6 \longrightarrow$ (ii)

$c - d = 10 \longrightarrow$ (iii)

$a - b = 2 \longrightarrow$ (iv)

Add $\longrightarrow$ (i) and $\longrightarrow$ (iv)

$$\begin{array}{l} a + \cancel{b} = 4 \\ a - \cancel{b} = 2 \\ \hline 2a = 6 \end{array}$$

$2a = 6$

Divide 2 on both sides

$$\frac{\cancel{2}a}{\cancel{2}} = \frac{\cancel{6}^{3}}{\cancel{2}_{1}}$$

$\boxed{a = 3}$

Using $\longrightarrow$ (i) Put $a = 3$

$a + b = 4 \longrightarrow$ (i)

$3 + b = 4$

$b = 4 - 3$

$\boxed{b = 1}$

Add $\longrightarrow$ (i) and $\longrightarrow$ (iv)

$$\begin{array}{l} c + \cancel{d} = 6 \\ c - \cancel{d} = 10 \\ \hline 2c = 16 \end{array}$$

$2c = 16$

Divide 2 on both sides

$$\frac{\cancel{2}c}{\cancel{2}} = \frac{\cancel{16}^{8}}{\cancel{2}_{1}}$$

$\boxed{c = 8}$

Using $\longrightarrow$ (ii) Put $c = 8$

$c + d = 6 \longrightarrow$ (ii)

$8 + d = 6$

$d = 6 - 8$

$\boxed{d = -2}$

- **Operations on Matrices:-**

Perform the following operations.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}, D = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

a. $C + E = E + C$

$Sol:-$

$$C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix} \quad E = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$C + E = ?$

$E + C = ?$

$$C + E = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix} \qquad E + C = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

$$C + E = \begin{bmatrix} 5 & -5 & 8 \\ 4 & 2 & 9 \\ 5 & 3 & 4 \end{bmatrix} \longrightarrow \text{(i)} \qquad E + C = \begin{bmatrix} 5 & -5 & 8 \\ 4 & 2 & 9 \\ 5 & 3 & 4 \end{bmatrix} \longrightarrow \text{(ii)}$$

From (i) & (ii) it is Proved that $\boxed{C + E = E + C}$

b. $A + B$

$Sol: -$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$$

Addition are not possible because the order of matrices is not same.

d. $-3C + 5O$

$Sol: -$

$-3C + 5O = ?$

$$C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Multiply -3 with matrix C

$$-3C = \begin{bmatrix} -9 & 3 & -9 \\ -12 & -3 & -15 \\ -6 & -3 & -9 \end{bmatrix}$$

Multiply 5 with matrix O

$$5O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-3C + 5O = \begin{bmatrix} -9 & 3 & -9 \\ -12 & -3 & -15 \\ -6 & -3 & -9 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{-3C + 5O = \begin{bmatrix} -9 & 3 & -9 \\ -12 & -3 & -15 \\ -6 & -3 & -9 \end{bmatrix}} \text{ Ans.}$$

f. $3D + 2F$

c. $D - F$

$Sol: -$

$$D = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}, F = \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix}$$

$D - F = ?$

$$D - f = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix}$$

$$\boxed{D - f = \begin{bmatrix} -1 & -7 \\ 0 & 1 \end{bmatrix}} \text{ Ans.}$$

e. $2B + F$

$Sol: -$

$$B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix}$$

Multiply 2 with matrix B

$$2B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \\ 6 & 4 \end{bmatrix}$$

Addition is not possible because

the order of matrices are not same.

Multiply 2 with matrix F

$$2F = \begin{bmatrix} -8 & 10 \\ 4 & 6 \end{bmatrix}$$

$3D + 2F = ?$

$Sol: -$

$D = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}$, $F = \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix}$

Multiply 3 with matrix D

$3D = \begin{bmatrix} 9 & -6 \\ 6 & 12 \end{bmatrix}$

$3D + 2F = \begin{bmatrix} 9 & -6 \\ 6 & 12 \end{bmatrix} + \begin{bmatrix} -8 & 10 \\ 4 & 6 \end{bmatrix}$

$\boxed{3D + 2F = \begin{bmatrix} 1 & 4 \\ 10 & 18 \end{bmatrix}}$ $Ans.$

- **Transpose:-**

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$

Show that $(A^t)^t = A$

Taking Transpose

$A^t = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{bmatrix} \longrightarrow$ (i)

Taking again Transpose

$(A^t)^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \longrightarrow$ (ii)

From $\longrightarrow$ (i) $\longrightarrow$ (ii) it is Proved that

$\boxed{(A^t)^t = A}$

2. $C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}$, $E = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

Show that $(C + E)^t = C^t + E^t$

Find $(C + E)^t = ?$

$C + E = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

Taking Transpose

$C^t = \begin{bmatrix} 3 & 4 & 2 \\ -1 & 1 & 1 \\ 3 & 5 & 3 \end{bmatrix} \longrightarrow$ (A)

$E = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

Taking Transpose

$E^t = \begin{bmatrix} 2 & 0 & 3 \\ -4 & 1 & 2 \\ 5 & 4 & 1 \end{bmatrix} \longrightarrow$ (B)

Add $\longrightarrow$ (A) & $\longrightarrow$ (B)

$(C + E)^t = \begin{bmatrix} 3 & 4 & 2 \\ -1 & 1 & 1 \\ 3 & 5 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 3 \\ -4 & 1 & 2 \\ 5 & 4 & 1 \end{bmatrix}$

$(C + E)^t = \begin{bmatrix} 5 & 4 & 5 \\ -5 & 2 & 3 \\ 8 & 9 & 4 \end{bmatrix} \longrightarrow$ (ii)

From $\longrightarrow$ (i) $\longrightarrow$ (ii) it is Proved that

$\boxed{(C + E)^t = C^t + E^t}$

$$C + E = \begin{bmatrix} 5 & -5 & 8 \\ 4 & 2 & 9 \\ 5 & 3 & 4 \end{bmatrix}$$

Taking Transpose

$$(C + E)^t = \begin{bmatrix} 5 & 4 & 5 \\ -5 & 2 & 3 \\ 8 & 9 & 4 \end{bmatrix} \longrightarrow \text{(i)}$$

Now Find $C^t + E^t = ?$

$$C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

- **Solution of System of Equation by Matrix Method:-**
  1) Gauss Elimination Method (Echelon Form)
  2) Gauss-Jordan Method (Reduced Echelon Form)
  3) By Matrix Inversion Method
  4) By Cramer's Rule

- **Multiplication:-**

1. $A = [\,1 \quad 2\,]$ $B = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

   Find $A.B$ and $B.A$

   $Sol: -$

   $A = [\,1 \quad 2\,]$, $B = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

   $A.B = ?$

   $A.B = [\,1 \quad 2\,].\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

   $A.B = [(1 \times 4) + (2 \times -1)]$

   $A.B = [4 - 2]$

   $\boxed{A.B = [2]}$

   $B.A = ?$

   $B.A = \begin{bmatrix} -4 \\ 1 \end{bmatrix}.[\,1 \quad 2\,]$

   $B.A = \begin{bmatrix} (4 \times 1) & (4 \times 2) \\ (-1 \times 1) & (-1 \times 2) \end{bmatrix}$

   $\boxed{B.A = \begin{bmatrix} 4 & 8 \\ -1 & -2 \end{bmatrix}}$

Compare

$13 + x^2 = 17$

$x^2 = 17 - 13$

$x^2 = 4$

Taking Square Root on Both Sides

$\sqrt{x^2} = \sqrt{4}$

$\boxed{x = 2}$ Ans

3. $A = [\,\cos\theta \quad \sin\theta\,]$ $B = \begin{bmatrix} Cos\,\theta \\ sin\,\theta \end{bmatrix}$ Find $A.B.$

   $Sol: -$

   $A = [\,\cos\theta \quad \sin\theta\,]$, $B = \begin{bmatrix} Cos\,\theta \\ sin\,\theta \end{bmatrix}$

   $A.B = ?$

   $A.B = [\,\cos\theta \quad \sin\theta\,].\begin{bmatrix} Cos\,\theta \\ sin\,\theta \end{bmatrix}$

   $A.B = [\,(\cos\theta \times \cos\theta) + (\,\sin\theta \times \sin\theta)\,]$

   $A.B = [\,\cos^2\theta + \sin^2\theta\,]$

   We know that $\cos^2\theta + \sin^2\theta = 1$

   $\boxed{A.B = 1}$

4. Find the value x so that $V_1.V_2 = 1.$

   $V_1 = [\,{}^{1}/_{2} \quad -1/2 \;\; x\,]$ $B = \begin{bmatrix} 1/2 \\ -1/2 \\ x \end{bmatrix}$

   $Sol: -$

   $V_1.V_2 = 1$

2. $A = [-3 \quad 2 \quad x\,]$ $B = \begin{bmatrix} -3 \\ 2 \\ x \end{bmatrix}$. If

$A.B = 17$ Find x ?

Sol: –

$A.B = 17$

$$[-3 \quad 2 \quad x\,] \,.\, \begin{bmatrix} -3 \\ 2 \\ x \end{bmatrix} = 17$$

$$[\,(-3 \times -3) + (2 \times 2) + (x \times x)\,] = 17$$

$$[\,9 + 4 + x^2\,] = 17$$

$$[\,13 + x^2\,] = 17$$

$$x^2 = \frac{\cancel{2}^{1}}{\cancel{4}_{2}}$$

Taking Square Root on Both Sides

$$\sqrt{x^{\cancel{2}}} = \sqrt{\frac{1}{2}}$$

$$\boxed{x = \pm\frac{1}{\sqrt{2}}}\ Ans$$

$$[\,{}^1/_2 \quad -{}^1/_2 \quad x\,] . \begin{bmatrix} 1/2 \\ -1/2 \\ x \end{bmatrix} = 1$$

$$[\,({}^1/_2 \times {}^1/_2) + (-{}^1/_2 \times -{}^1/_2) + (x \times x)\,] = 1$$

$$[\,({}^1/_4) + ({}^1/_4) + x^2\,] = 1$$

$${}^1/_4 + {}^1/_4 + x^2 = 1$$

$${}^1/_4 + x^2 = 1 - {}^1/_4$$

$${}^1/_4 + x^2 = \frac{4-1}{4} \;=>\; x^2 = \frac{3}{4} - \frac{1}{4} \;=>\; x^2 = \frac{3-1}{4}$$

$$x^2 = \frac{2}{4}$$

$$\boxed{y = \frac{12}{5}} \quad \text{Put } y = \frac{12}{5} \text{ in} \longrightarrow \text{(ii)}$$

$$-x + 3(\frac{12}{5}) = 6 \;=>\; -x + (\frac{36}{5}) = 6$$

$$=> -x = 6 - \frac{36}{5} \;=>\; -x = \frac{30-36}{5}$$

$$-x = -\frac{6}{5} \quad => \; \cancel{-}x = \cancel{-}\frac{6}{5}$$

$$\boxed{x = \frac{6}{5}}\ Ans$$

5. Let $A = \begin{bmatrix} 1 & 2 & x \\ 3 & -1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$

$A.B = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$ Find x and .

Sol: –

$A.B = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & x \\ 3 & -1 & 2 \end{bmatrix} . \begin{bmatrix} y \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} (1 \times y) + (2 \times x) + (x \times 1) \\ (3 \times y) + (-1 \times x) + (2 \times 1) \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} y + 2x + x \\ 3y - x + 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} y + 3x \\ 3y - x + 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

Compare

$3x + y = 6$

$-x + 3y + 2 = 8$

$3x + y = 6 \longrightarrow$ (i)

$-x + 3y = 6 \longrightarrow$ (ii)

Multiply $\longrightarrow$ (ii) by 3

$-3x + 9y = 18 \longrightarrow$ (ii)

Add $\longrightarrow$ (ii) and $\longrightarrow$ (i)

$\cancel{3x} + y = 6$

$-\cancel{3x} + 9y = 18$

$0 + 10y = 24$

$10y = 24$

$y = \dfrac{\cancel{24}^{12}}{\cancel{10}^{5}}$

Properties of Transpose:-

- **Symmetric Property:-**

A square matrix will be symmetric if $A^t = A$. Symmetric property will be occurs

Only in square matrix.

- **Examples:-**

$\rightarrow A = \begin{bmatrix} 2 & 7 \\ 7 & 4 \end{bmatrix}$......... (i)

Taking Transpose

$A^t = \begin{bmatrix} 2 & 7 \\ 7 & 4 \end{bmatrix}$......... (ii)

From (i) & (ii) it is proved that

$\boxed{A^t = A}$

Hence it is Symmetric.

Hence it is Symmetric.

ii. Show that **(A+B)** is symmetric.

$$A + B = \begin{bmatrix} 3 & 7 & 4 \\ 7 & 10 & 9 \\ 4 & 9 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 7 & 10 \\ 7 & 2 & 3 \\ 10 & 3 & 15 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 3+3 & 7+7 & 4+10 \\ 7+7 & 10+2 & 9+3 \\ 4+10 & 9+3 & -2+15 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 9 & 14 & 14 \\ 14 & 12 & 12 \\ 14 & 12 & 13 \end{bmatrix} \text{......... (i)}$$

Taking Transpose

$$(A + B)^t = \begin{bmatrix} 9 & 14 & 14 \\ 14 & 12 & 12 \\ 14 & 12 & 13 \end{bmatrix} \text{......... (ii)}$$

From (i) & (ii) it is proved that

$\boxed{(A + B)^t = (A + B)}$

- **Questions:-**

1. If $A = \begin{bmatrix} 3 & 7 & 4 \\ 7 & 10 & 9 \\ 4 & 9 & -2 \end{bmatrix}, B = \begin{bmatrix} 3 & 7 & 10 \\ 7 & 2 & 3 \\ 10 & 3 & 15 \end{bmatrix}$ then
   i. Show that **A** and **B** are symmetric.
   ii. Show that **(A+B)** is symmetric.

$Sol: -$

i. Show that **A** and **B** are symmetric

2. If $A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 7 & 8 \\ 2 & 4 & 6 \end{bmatrix}$ then
   i. Show that $\boldsymbol{A + A^t}$ is Symmetric

$Sol: -$

$$A = \begin{bmatrix} 3 & 7 & 4 \\ 7 & 10 & 9 \\ 4 & 9 & -2 \end{bmatrix}$$

Taking Transpose

$$A^t = \begin{bmatrix} 3 & 7 & 4 \\ 7 & 10 & 9 \\ 4 & 9 & -2 \end{bmatrix} \text{............ (i)}$$

$$B = \begin{bmatrix} 3 & 7 & 10 \\ 7 & 2 & 3 \\ 10 & 3 & 15 \end{bmatrix}$$

Taking Transpose

$$B^t = \begin{bmatrix} 3 & 7 & 10 \\ 7 & 2 & 3 \\ 10 & 3 & 15 \end{bmatrix} \text{............ (ii)}$$

From **(i)** and **(ii)** it is proved that **A** and **B** are symmetric

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 7 & 8 \\ 2 & 4 & 6 \end{bmatrix}$$

Taking Transpose

$$A^t = \begin{bmatrix} 3 & 5 & 2 \\ 1 & 7 & 4 \\ 2 & 8 & 6 \end{bmatrix}$$

Add A and A^t

$$A + A^t = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 7 & 8 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 2 \\ 1 & 7 & 4 \\ 2 & 8 & 6 \end{bmatrix}$$

$$A + A^t = \begin{bmatrix} 3+3 & 1+5 & 2+2 \\ 5+1 & 7+7 & 8+4 \\ 2+2 & 4+8 & 6+6 \end{bmatrix}$$

$$A + A^t = \begin{bmatrix} 6 & 6 & 4 \\ 6 & 14 & 12 \\ 4 & 12 & 12 \end{bmatrix} \text{............ (i)}$$

Taking Transpose

$$(A + A^t)^t = \begin{bmatrix} 6 & 6 & 4 \\ 6 & 14 & 12 \\ 4 & 12 & 12 \end{bmatrix} \text{............ (ii)}$$

From **(i)** and **(ii)** it is proved that

$$\boxed{A + A^t = (A + A^t)^t}$$

Hence it is Symmetric.

3. $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

Show that $(\boldsymbol{A} + \boldsymbol{A^t})$ is symmetric.

Sol: –

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Taking Transpose

$$A^t = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

Add A and A^t

$$A + A^t = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$A + A^t = \begin{bmatrix} a+a & b+d & c+g \\ b+d & e+e & f+h \\ c+g & f+h & i+i \end{bmatrix}$$

$$A + A^t = \begin{bmatrix} 2a & b+d & c+g \\ b+d & 2e & f+h \\ c+g & f+h & 2i \end{bmatrix} \text{............ (i)}$$

Taking Transpose

$$(A + A^t)^t = \begin{bmatrix} 2a & b+d & c+g \\ b+d & 2e & f+h \\ c+g & f+h & 2i \end{bmatrix} \text{............ (ii)}$$

From **(i)** and **(ii)** it is proved that $\boldsymbol{A + A^t}$ is symmetric.

- **Skew-Symmetric Property:-**

A square matrix will be Skew-symmetric if $A^t = -A$.The Skew-symmetric property will be occurs in those matrices whose diagonal is zero.

- **Example:-**

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$Sol: -$

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \text{............ (i)}$$

Taking Transpose

$$A^t = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

Taking $(-)$ Common

$$A^t = -\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \text{............ (ii)}$$

From (i) & (ii) it is proved that

$$\boxed{A^t = -A}$$

Hence it is Skew-symmetric

- **Questions:-**

1. $A = \begin{bmatrix} 0 & 3 & -2 \\ -3 & 0 & 5 \\ 2 & -5 & 0 \end{bmatrix}$

$Sol: -$

$$A - B = \begin{bmatrix} 3 & 7 & 4 \\ 7 & 10 & 9 \\ 4 & 9 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 7 & 10 \\ 7 & 2 & 3 \\ 10 & 3 & 15 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 3-3 & 7-7 & 4-10 \\ 7-7 & 10-2 & 9-3 \\ 4-10 & 9-3 & -2-15 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 0 & 0 & -6 \\ 0 & 8 & 6 \\ -6 & 6 & -17 \end{bmatrix} \text{............ (i)}$$

Taking Transpose

$$(A - B)^t = \begin{bmatrix} 0 & 0 & -6 \\ 0 & 8 & 6 \\ -6 & 6 & -17 \end{bmatrix}$$

Taking $(-)$ Common

$$(A - B)^t = -\begin{bmatrix} 0 & 0 & 6 \\ 0 & -8 & -6 \\ 6 & -6 & 17 \end{bmatrix} \text{............ (ii)}$$

From (i) & (ii) it is proved that

$$A = \begin{bmatrix} 0 & 3 & -2 \\ -3 & 0 & 5 \\ 2 & -5 & 0 \end{bmatrix} \quad \text{............ (i)}$$

Taking Transpose

$$A^t = \begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & -5 \\ -2 & 5 & 0 \end{bmatrix}$$

Taking $(-)$ Common

$$A^t = -\begin{bmatrix} 0 & 3 & -2 \\ -3 & 0 & 5 \\ 2 & -5 & 0 \end{bmatrix} \quad \text{............ (ii)}$$

From (i) & (ii) it is proved that

$$\boxed{A^t = -A}$$

Hence it is Skew-symmetric

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2. $A = \begin{bmatrix} 3 & 7 & 4 \\ 7 & 10 & 9 \\ 4 & 9 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 7 & 10 \\ 7 & 2 & 3 \\ 10 & 3 & 15 \end{bmatrix}$

ii. Show $(\boldsymbol{A} - \boldsymbol{B})$ is Skew-symmetric

$Sol: -$

$$A = \begin{bmatrix} 3 & 7 & 4 \\ 7 & 10 & 9 \\ 4 & 9 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 7 & 10 \\ 7 & 2 & 3 \\ 10 & 3 & 15 \end{bmatrix}$$

3. $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

i. Show that $A - A^t$ is Skew-symmetric.

$Sol: -$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Taking Transpose

$$A^t = \begin{bmatrix} 3 & 4 & 7 \\ 2 & 5 & 8 \\ 1 & 6 & 9 \end{bmatrix}$$

Subtract $\boldsymbol{A}$ from $\boldsymbol{A^t}$

$$A - A^t = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 4 & 7 \\ 2 & 5 & 8 \\ 1 & 6 & 9 \end{bmatrix}$$

$$A - A^t = \begin{bmatrix} 3-3 & 2-4 & 1-7 \\ 4-2 & 5-5 & 6-8 \\ 7-1 & 8-6 & 9-9 \end{bmatrix}$$

$$\boxed{(A - B)^t \neq -(A - B)}$$

Hence it is not Skew-symmetric

Taking Transpose

$$A^t = \begin{bmatrix} \mathrm{a} & \mathrm{d} & \mathrm{g} \\ \mathrm{b} & \mathrm{e} & \mathrm{h} \\ \mathrm{c} & \mathrm{f} & \mathrm{i} \end{bmatrix}$$

Subtract $\boldsymbol{A}$ from $\boldsymbol{A^t}$

$$A - A^t = \begin{bmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{d} & \mathrm{e} & \mathrm{f} \\ \mathrm{g} & \mathrm{h} & \mathrm{i} \end{bmatrix} - \begin{bmatrix} \mathrm{a} & \mathrm{d} & \mathrm{g} \\ \mathrm{b} & \mathrm{e} & \mathrm{h} \\ \mathrm{c} & \mathrm{f} & \mathrm{i} \end{bmatrix}$$

$$A - A^t = \begin{bmatrix} \mathrm{a-a} & \mathrm{b-d} & \mathrm{c-g} \\ \mathrm{d-b} & \mathrm{e-e} & \mathrm{f-h} \\ \mathrm{g-c} & \mathrm{h-f} & \mathrm{i-i} \end{bmatrix}$$

$$A - A^t = \begin{bmatrix} 0 & \mathrm{b-d} & \mathrm{c-g} \\ \mathrm{d-b} & 0 & \mathrm{f-h} \\ \mathrm{g-c} & \mathrm{h-f} & 0 \end{bmatrix} \quad \text{............ (i)}$$

$$A - A^t = \begin{bmatrix} 0 & -2 & -6 \\ 2 & 0 & -2 \\ 6 & 2 & 0 \end{bmatrix} \quad \text{........... (i)}$$

Taking Transpose

$$(A - A^t)^t = \begin{bmatrix} 0 & 2 & 6 \\ -2 & 0 & 2 \\ -6 & -2 & 0 \end{bmatrix}$$

Taking $(-)$ Common

$$(A - A^t)^t = -\begin{bmatrix} 0 & -2 & -6 \\ 2 & 0 & -2 \\ 6 & 2 & 0 \end{bmatrix} \quad \text{........... (ii)}$$

From (i) & (ii) it is proved that

$$\boxed{(A - A^t)^t = -(A - A^t)}$$

Hence it is Skew-symmetric

4. $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

i. Show that $(A - A^t)$ is Skew-symmetric

$Sol:-$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Taking Transpose

$$(A - A^t)^t = \begin{bmatrix} 0 & d-b & g-c \\ b-d & 0 & h-f \\ c-g & f-h & 0 \end{bmatrix}$$

Taking $(-)$ Common

$$(A - A^t)^t = -\begin{bmatrix} 0 & -(d-b) & -(g-c) \\ -(b-d) & 0 & -(h-f) \\ -(c-g) & -(f-h) & 0 \end{bmatrix}$$

$$(A - A^t)^t = -\begin{bmatrix} 0 & b-d & c-g \\ d-b & 0 & f-h \\ g-c & h-f & 0 \end{bmatrix} \quad \text{........... (ii)}$$

From (i) & (ii) it is proved that

$$\boxed{(A - A^t)^t = -(A - A^t)}$$

Hence it is Skew-symmetric

- **Using Gauss Elimination method(Echelon Form) .Find all solution of system of equation:-**

1. $x + y + 2z = -1$
   $x - 2y + z = -5$
   $3x + y + z = 3$

   $Sol:-$

Matrix Form

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 3 \end{bmatrix}$$

$$\quad A \qquad\qquad X \qquad B$$

<u>Gauss Elimination method</u>

[A/B] Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & -2 & 1 & -5 \\ 3 & 1 & 1 & 3 \end{array}\right] \sim R_2 - R_1$$

3rd Row

$0.x + 0.y - 26z = 52$

$0 + 0 - 26z = 52$

$-26z = 52$

$$z = \frac{-52}{26}$$

$$\boxed{z = -2}$$

2nd Row

$0.x + 1.y + 9z = -16$

Put $z = -2$

$y + 9(-2) = -16$

$y - 18 = -16$

$y = -16 + 18$

$$\boxed{y = 2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1-1 & -2-1 & 1-2 & -5+1 \\ 3 & 1 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 3 & 1 & 1 & 3 \end{array}\right] \sim R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 3-3 & 1-3 & 1-6 & 3+3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 0 & -2 & -5 & 6 \end{array}\right] -2R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 0 & 4 & 10 & -12 \end{array}\right] \sim R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3+4 & -1+10 & -4-12 \\ 0 & 4 & 10 & -12 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 9 & -16 \\ 0 & 4 & 10 & -12 \end{array}\right] \sim R_3 - 4R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 9 & -16 \\ 0 & 4-4 & 10-36 & -12+64 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 9 & -16 \\ 0 & 0 & -26 & 52 \end{array}\right]$$

Backward Substitution

1[st] Row

$1.x + 1.y + 2z = -1$

$x + y + 2z = -1$

Put $y = 2$ & $z = -2$

$x + 2 + 2(-2) = -1$

$x + 2 - 4 = -1$

$x - 2 = -1$

$x = -1 + 2$

$\boxed{x = 1}$

2. $2x + 4y + 6z = -12$
$2x - 3y - 4z = 15$
$3x + 4y + 5z = -8$

$Sol: -$

Matrix Form

$$\begin{bmatrix} 2 & 4 & 6 \\ 2 & -3 & -4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -12 \\ 15 \\ -8 \end{bmatrix}$$

$A \quad X \quad B$

<u>Gauss Elimination method</u>

[A/B] Augmented Matrix

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & -12 \\ 2 & -3 & -4 & 15 \\ 3 & 4 & 5 & -8 \end{array}\right] \sim R_2 - R_1$$

$3rd\ Row$

$0.x + 0.y - 16z = 32$

$0 + 0 - 16z = 32$

$-16z = 32$

$y = -\frac{28^1}{28}$

$\boxed{z = -2}$

2[nd] Row

$0.x + 28.y + 40.z = -108$

Put $z = -2$

$28y + 40(-2) = -108$

$28y - 80 = -108$

$28y = -108 + 80$

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & -12 \\ 2-2 & -3-4 & -4-6 & 15\ +12 \\ 3 & 4 & 5 & -8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & -12 \\ 0 & -7 & -10 & 27 \\ 3 & 4 & 5 & -8 \end{array}\right] \begin{matrix} -3R_1 \\ 2R_3 \end{matrix}$$

$$\left[\begin{array}{ccc|c} -6 & -12 & -18 & 36 \\ 0 & -7 & -10 & 27 \\ 6 & 8 & 10 & -16 \end{array}\right] \sim R_1 + R_3$$

$$\left[\begin{array}{ccc|c} -6 & -12 & -18 & 36 \\ 0 & -7 & -10 & 27 \\ 6-6 & 8-12 & 10-18 & -16+36 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -6 & -12 & -18 & 36 \\ 0 & -7 & -10 & 27 \\ 0 & -4 & -8 & 20 \end{array}\right] \begin{matrix} -4R_2 \\ 7R_3 \end{matrix}$$

$$\left[\begin{array}{ccc|c} -6 & -12 & -18 & 36 \\ 0 & 28 & 40 & -108 \\ 0 & -28 & -56 & 140 \end{array}\right] \sim R_2 + R_3$$

$$\left[\begin{array}{ccc|c} -6 & -12 & -18 & 36 \\ 0 & 28 & 40 & -108 \\ 0 & 28-28 & 40-56 & -108+140 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -6 & -12 & -18 & 36 \\ 0 & 28 & 40 & -108 \\ 0 & 0 & -16 & 32 \end{array}\right]$$

Backward Substitution

$28y = -28$

$$x = \frac{\cancel{-12}^{\,2}}{\cancel{-6}}$$

$\boxed{y = -1}$

1st Row

$-6x - 12y - 18z = 36$

Put $y = -1$ & $z = -2$

$-6x - 12(-1) - 18(-2) = 36$

$-6x + 12 + 36 = 36$

$-6x + 48 = 36$

$-6x = 36 - 48$

$-6x = -12$

$$x = \frac{\cancel{-12}^{\,2}}{\cancel{-6}}$$

$\boxed{x = 2}$

- **Using Gauss Jordan method(Reduce Echelon Form) .Find all solution of system of equation:-**

1. $x + y + 2z = -1$
   $x - 2y + z = -5$
   $3x + y + z = 3$

   Sol: –

   Matrix Form

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 3 \end{bmatrix}$$

$$\quad A \qquad\qquad X \qquad\quad B$$

<u>Gauss Jordan method</u>

[A/B] Augmented Matrix

Now Convert into Reduce Echelon Form

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 9 & -16 \\ 0 & 0 & -26 & 52 \end{array}\right] \sim R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & -2 & 1 & -5 \\ 3 & 1 & 1 & 3 \end{array}\right] \sim R_2 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1-1 & -2-1 & 1-2 & -5+1 \\ 3 & 1 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 3 & 1 & 1 & 3 \end{array}\right] \sim R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 3-3 & 1-3 & 1-6 & 3+3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 0 & -2 & -5 & 6 \end{array}\right] -2R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 0 & 4 & 10 & -12 \end{array}\right] \sim R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3+4 & -1+10 & -4\ -12 \\ 0 & 4 & 10 & -12 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 9 & -16 \\ 0 & 4 & 10 & -12 \end{array}\right] \sim R_3 - 4R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 9 & -16 \\ 0 & 4-4 & 10-36 & -12+64 \end{array}\right]$$

Echelon Form

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 9 & -16 \\ 0 & 0 & -26 & 52 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1-0 & 1-1 & 2-9 & -1+16 \\ 0 & 1 & 9 & -16 \\ 0 & 0 & -26 & 52 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & 15 \\ 0 & 1 & 9 & -16 \\ 0 & 0 & -26 & 52 \end{array}\right] \sim \frac{R_3}{-26}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & 15 \\ 0 & 1 & 9 & -16 \\ 0 & 0 & 1 & \frac{52}{-26} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & 15 \\ 0 & 1 & 9 & -16 \\ 0 & 0 & 1 & -2 \end{array}\right] \sim R_2 - 9R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & 15 \\ 0-0 & 1-0 & 9-9 & -16+18 \\ 0 & 0 & 1 & -2 \end{array}\right] \sim R_2 - 9R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & 15 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \sim R_1 + 7R_3$$

$$\left[\begin{array}{ccc|c} 1+0 & 0+0 & -7+7 & 15-14 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$\boxed{x = 1}$

$\boxed{y = 2}$

$\boxed{z = -2}$

- **Solution of Three equation and three variables through matrix:-**

1. $x + y + 2z + 3w = 13$

   $x - 2y + z + w = 8$

   $3x + y + z - w = 1$

   $Sol: -$

   $Matrix\ Form$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & -2 & 1 & 1 \\ 3 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 13 \\ 8 \\ 1 \end{bmatrix}$$

$A \qquad X \qquad B$

[A/B] Augmented Matrix

$z = 8 - 2w$

Put $w = r$

$\boxed{z = 8 - 2r}$

2[nd] Row

$0.x - 6y - 2z - 4w = -10$

$-6y - 2z - 4w = -10$

Put $z = 8 - 2r$ & $w = r$

$-6y - 2(8 - 2r) - 4(r) = -10$

$-6y - 16 + \cancel{4r} - \cancel{4r} = -10$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 1 & -2 & 1 & 1 & 8 \\ 3 & 1 & 1 & -1 & 1 \end{array}\right] \sim R_2 - R_1$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 1-1 & -2-1 & 1-2 & 1-3 & 8-13 \\ 3 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 0 & -3 & -1 & -2 & -5 \\ 3 & 1 & 1 & -1 & 1 \end{array}\right] \sim R_3 - 3R_1$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 0 & -3 & -1 & -2 & -5 \\ 3-3 & 1-3 & 1-6 & -1-9 & 1-39 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 0 & -3 & -1 & -2 & -5 \\ 0 & -2 & -5 & -10 & -38 \end{array}\right] \begin{array}{c} 2R_2 \\ -3R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 0 & -6 & -2 & -4 & -10 \\ 0 & 6 & 15 & 30 & 144 \end{array}\right] \sim R_3 + R_2$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 0 & -6 & -2 & -4 & -10 \\ 0 & 6-6 & 15-2 & 30-4 & 144-10 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 0 & -6 & -2 & -4 & -10 \\ 0 & 0 & 13 & 26 & 104 \end{array}\right]$$

Backward Substitution

$3rd\ Row$

$0.x + 0.y + 13z + 26w = 104$

$13z + 26w = 104$

$13z = 104 - 26w$

Dividing 13 on both sides

$$\frac{\cancel{13}z}{\cancel{13}} = \frac{\cancel{104}^{8}}{\cancel{13}} - \frac{\cancel{26}^{2}w}{\cancel{13}}$$

$-6y - 16 = -10$

$-6y = -10 + 16 \quad => \ -6y = 6$

Dividing -6 on both sides

$$\frac{\cancel{-6}y}{\cancel{-6}} = \frac{\cancel{6}}{\cancel{-6}}$$

$\boxed{y = -1}$

1st Row

$1.x + 1.y + 2z + 3w = 13$

Put $z = 8 - 2r$, $w = r$ & $y = -1$

$x + (-1) + 2(8 - 2r) + 3(r) = 13$

$x - 1 + 16 - 4r + 3r = 13$

$x + 15 - r = 13$

$x = 13 - 15 + r$

$\boxed{x = -2 + r}$