Discrete Sturcture

Set Theory

Actually, you will see that logic and set theory are very closely related.

Set Theory

- Set: A well defined unordered collection of distinct elements is called a set.
- Set: Collection of objects (called elements)
- $a \in A$ "a is an element of A"
  "a is a member of A"
- $a \notin A$ "a is not an element of A"
- $A = \{a_1, a_2, \ldots, a_n\}$ "A contains $a_1, \ldots, a_n$"
- Order of elements is insignificant

Set Equality

Sets A and B are equal if and only if they contain exactly the same elements.

Examples:

- $A = \{9, 2, 7, -3\}$, $B = \{7, 9, -3, 2\}$: $A = B$
- $A = \{dog, cat, horse\}$,
  $B = \{cat, horse, squirrel, dog\}$: $A \neq B$
- $A = \{dog, cat, horse\}$,
  $B = \{cat, horse, dog, dog\}$: $A = B$

Examples for Sets

"Standard" Sets:

- Natural numbers **N** = {0, 1, 2, 3, …}
- Integers **Z** = {…, -2, -1, 0, 1, 2, …}
- Positive Integers $\mathbf{Z}^+$ = {1, 2, 3, 4, …}
- Real Numbers **R** = {47.3, -12, π, …}
- Rational Numbers **Q** = {1.5, 2.6, -3.8, 15, …}

(correct definitions will follow)

Examples for Sets

- Null set/Empty set: A set with no elements.
- Denoted as $\varnothing$, {}
- $A = \varnothing$ "empty set/null set"

Subsets

Subsets: If every element of Set A is also element of set B Then A is subset of B.

$A \subseteq B$ "A is a subset of B"

$A \subseteq B$ if and only if every element of A is also an element of B.

Examples:

$A = \{3, 9\}, B = \{5, 9, 1, 3\},$ $A \subseteq B$? true

$A = \{3, 3, 3, 9\}, B = \{5, 9, 1, 3\},$ $A \subseteq B$? true

$A = \{1, 2, 3\}, B = \{2, 3, 4\},$ $A \subseteq B$? false

Subsets

Proper subsets:

Let A and B be sets. A is the proper subset of B. IF and only if every element of A is in B but there is at least one element of B that is not in A.

$A \subset B$ “A is a proper subset of B”

Example:

Let A={1,3,5} B= {1,2,3,5}

Cardinality of Sets

Cardinality: Total no of elements in a set.

Examples:

A = {Mercedes, BMW, Alto}, $|A| = 3$

B = {1, {2, 3}, {4, 5}, 6} $|B| = 4$

C = $\varnothing$ $|C| = 0$

The Power Set

Power set: If 'A' is finite set then set of all subsets of A is called power set of 'A'.

Examples:

$A = \{x, y, z\}$

$P(A) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$

Cartesian Product

The ordered n-tuple $(a_1, a_2, a_3, \ldots, a_n)$ is an ordered collection of n objects.

Two ordered n-tuples $(a_1, a_2, a_3, \ldots, a_n)$ and $(b_1, b_2, b_3, \ldots, b_n)$ are equal if and only if they contain the same elements in the same order.

The Cartesian product of two sets is defined as:

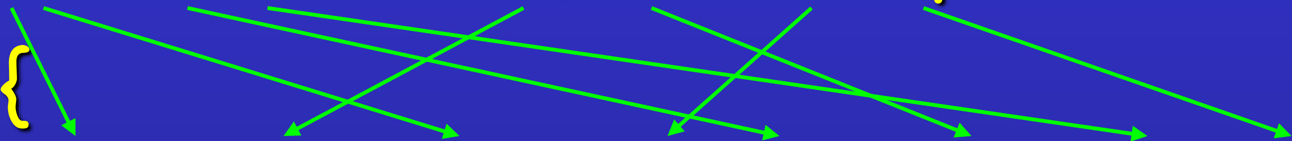
$A \times B = \{(a, b) \mid a \in A \land b \in B\}$

Cartesian Product

Example:

A = {good, bad}, B = {student, prof}

A×B = {



(good, student), (good, prof), (bad, student), (bad, prof)}

B×A = {(student, good), (prof, good), (student, bad), (prof, bad)}

Example: A = {x, y}, B = {a, b, c}

A×B = {(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)}

Set Operations

Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$

$A \cup B = \{a, b, c, d\}$

Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$

$A \cap B = \{b\}$

Set Operations

Two sets are called disjoint if their intersection is empty, that is, they share no elements:

$A \cap B = \emptyset$

The difference between two sets A and B contains exactly those elements of A that are not in B:

$A-B = \{x \mid x \in A \wedge x \notin B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$, $A-B = \{a\}$

Functions

Functions

A function f from a set A to a set B is an assignment of exactly one element of B to each element of A.

We write

f(a) = b

if b is the unique element of B assigned by the function f to the element a of A.

If f is a function from A to B, we write

$f: A \rightarrow B$

Functions

If $f:A \rightarrow B$, we say that A is the domain of f and B is the codomain of f.

If f(a) = b, we say that b is the image of a and a is the pre-image of b.

The range of $f:A \rightarrow B$ is the set of all images of all elements of A.

We say that $f:A \rightarrow B$ maps A to B.

Functions

P = {Linda, Max, Kathy, Peter}

C = {Boston, New York, Hong Kong, Moscow}

f(Linda) = Moscow

f(Max) = Boston

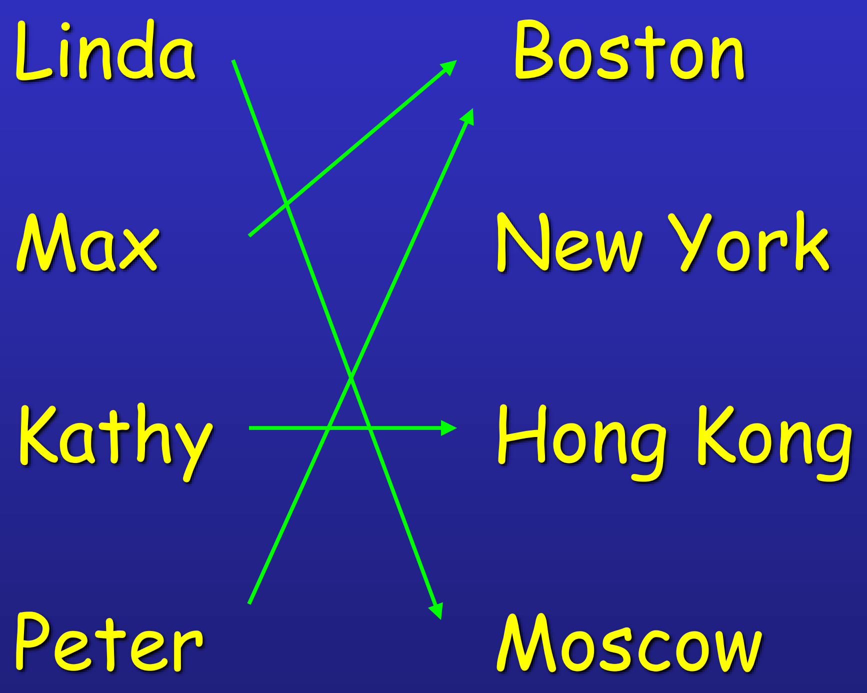
f(Kathy) = Hong Kong

f(Peter) = New York

Functions

Other ways to represent f:

x	f(x)
Linda	Moscow
Max	Boston
Kathy	Hong Kong
Peter	Boston



Properties of Functions

A function f is one-to-one if and only if it does not map two distinct elements of A onto the same element of B.

Properties of Functions

And again…

f(Linda) = Moscow	g(Linda) = Moscow
f(Max) = Boston	g(Max) = Boston
f(Kathy) = Hong Kong	g(Kathy) = Hong Kong
f(Peter) = Boston	g(Peter) = New York

Is f one-to-one?

No, Max and Peter are mapped onto the same element of the image.

Is g one-to-one?

Yes, each element is assigned a unique element of the image.

Properties of Functions

How can we prove that a function f is one-to-one?

Whenever you want to prove something, first take a look at the relevant definition(s):

$\forall x, y \in A \, (f(x) = f(y) \rightarrow x = y)$

Example:

$f: \mathbf{R} \rightarrow \mathbf{R}$

$f(x) = x^2$

Disproof by counterexample:

$f(3) = f(-3)$, but $3 \neq -3$, so f is not one-to-one.

Properties of Functions

… and yet another example:

$f: \mathbf{R} \to \mathbf{R}$

$f(x) = 3x$

One-to-one: $\forall x, y \in A\ (f(x) = f(y) \to x = y)$

To show: $f(x) \neq f(y)$ whenever $x \neq y$ (indirect proof)

$x \neq y$

$\Leftrightarrow 3x \neq 3y$

$\Leftrightarrow f(x) \neq f(y)$,

so if $x \neq y$, then $f(x) \neq f(y)$, that is, f is one-to-one.

Properties of Functions

A function $f:A \rightarrow B$ with $A,B \subseteq R$ is called strictly increasing, if

$\forall x,y \in A \ (x < y \rightarrow f(x) < f(y))$,

and strictly decreasing, if

$\forall x,y \in A \ (x < y \rightarrow f(x) > f(y))$.

Obviously, a function that is either strictly increasing or strictly decreasing is one-to-one.

Properties of Functions

A function $f:A \to B$ is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

In other words, f is onto if and only if its range is its entire codomain.

A function $f: A \to B$ is a one-to-one correspondence, or a bijection, if and only if it is both one-to-one and onto.

Obviously, if f is a bijection and A and B are finite sets, then $|A| = |B|$.

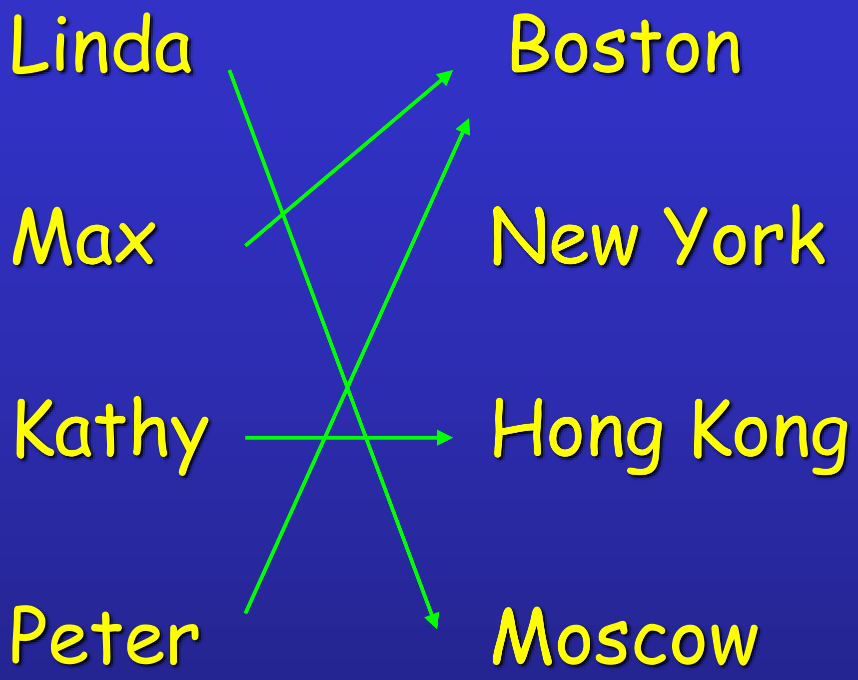
Properties of Functions

Examples:

In the following examples, we use the arrow representation to illustrate functions $f:A \rightarrow B$.

In each example, the complete sets A and B are shown.

Properties of Functions



Is f injective?

No.

Is f surjective?

No.

Is f bijective?

No.

Properties of Functions

Is f injective?

No.

Is f surjective?

Yes.

Is f bijective?

No.

Properties of Functions

Is f injective?

Yes.

Is f surjective?

No.

Is f bijective?

No.

Properties of Functions

Is f injective?

No! f is not even a function!

Properties of Functions

Is f injective?

Yes.

Is f surjective?

Yes.

Is f bijective?

Yes.

Inversion

An interesting property of bijections is that they have an **inverse function**.

The **inverse function** of the bijection $f: A \to B$ is the function $f^{-1}: B \to A$ with

$f^{-1}(b) = a$ whenever $f(a) = b$.

Inversion

Example:

$f(\text{Linda}) = \text{Moscow}$
$f(\text{Max}) = \text{Boston}$
$f(\text{Kathy}) = \text{Hong Kong}$
$f(\text{Peter}) = \text{Lübeck}$
$f(\text{Helena}) = \text{New York}$

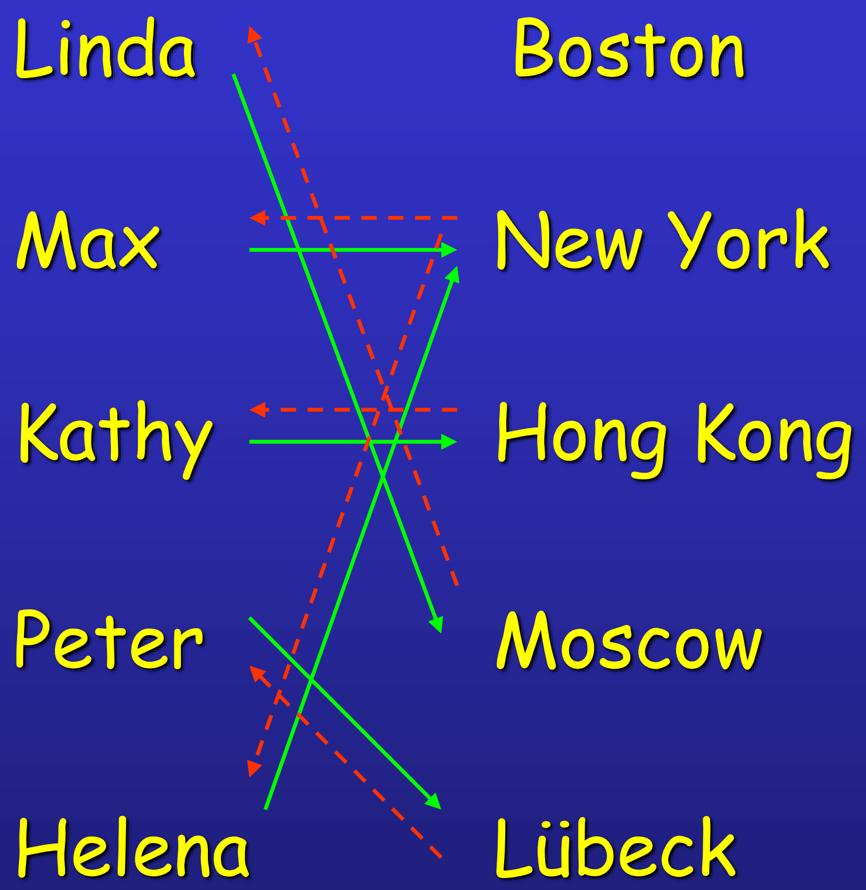
Clearly, f is bijective.

The inverse function f^{-1} is given by:

$f^{-1}(\text{Moscow}) = \text{Linda}$
$f^{-1}(\text{Boston}) = \text{Max}$
$f^{-1}(\text{Hong Kong}) = \text{Kathy}$
$f^{-1}(\text{Lübeck}) = \text{Peter}$
$f^{-1}(\text{New York}) = \text{Helena}$

Inversion is only possible for bijections (= invertible functions)

Inversion



$f^{-1}: C \rightarrow P$ is no function, because it is not defined for all elements of C and assigns two images to the pre-image New York.

Composition

The **composition** of two functions $g:A \rightarrow B$ and $f:B \rightarrow C$, denoted by $f \circ g$, is defined by

$(f \circ g)(a) = f(g(a))$

This means that

- **first**, function g is applied to element $a \in A$, mapping it onto an element of B,
- **then**, function f is applied to this element of B, mapping it onto an element of C.
- **Therefore**, the composite function maps from A to C.

Composition

Example:

$f(x) = 7x - 4$, $g(x) = 3x$,

$f:\mathbf{R}\to\mathbf{R}$, $g:\mathbf{R}\to\mathbf{R}$

$(f\circ g)(5) = f(g(5)) = f(15) = 105 - 4 = 101$

$(f\circ g)(x) = f(g(x)) = f(3x) = 21x - 4$

Composition

Composition of a function and its inverse:

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$

The composition of a function and its inverse is the **identity function** $i(x) = x$.

Graphs

The **graph** of a function $f:A \rightarrow B$ is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.

The graph is a subset of $A \times B$ that can be used to visualize f in a two-dimensional coordinate system.

Floor and Ceiling Functions

The **floor** and **ceiling** functions map the real numbers onto the integers ($\mathbf{R} \rightarrow \mathbf{Z}$).

The **floor** function assigns to $r \in \mathbf{R}$ the largest $z \in \mathbf{Z}$ with $z \leq r$, denoted by $\lfloor r \rfloor$.

Examples: $\lfloor 2.3 \rfloor = 2$, $\lfloor 2 \rfloor = 2$, $\lfloor 0.5 \rfloor = 0$, $\lfloor -3.5 \rfloor = -4$

The **ceiling** function assigns to $r \in \mathbf{R}$ the smallest $z \in \mathbf{Z}$ with $z \geq r$, denoted by $\lceil r \rceil$.

Examples: $\lceil 2.3 \rceil = 3$, $\lceil 2 \rceil = 2$, $\lceil 0.5 \rceil = 1$, $\lceil -3.5 \rceil = -3$

Now, something about

BooleanAl
gebra

Boolean Algebra

Boolean algebra provides the operations and the rules for working with the set **{0, 1}**.

These are the rules that underlie **electronic circuits**, and the methods we will discuss are fundamental to **VLSI design**.

We are going to focus on three operations:

- Boolean complementation,
- Boolean sum, and
- Boolean product

Boolean Operations

The **complement** is denoted by a bar (on the slides, we will use a minus sign). It is defined by

$-0 = 1$ and $-1 = 0$.

The **Boolean sum**, denoted by + or by OR, has the following values:

$1 + 1 = 1$, $1 + 0 = 1$, $0 + 1 = 1$, $0 + 0 = 0$

The **Boolean product**, denoted by · or by AND, has the following values:

$1 \cdot 1 = 1$, $1 \cdot 0 = 0$, $0 \cdot 1 = 0$, $0 \cdot 0 = 0$

Boolean Functions and Expressions

Definition: Let $B = \{0, 1\}$. The variable x is called a **Boolean variable** if it assumes values only from B.

A function from B^n, the set $\{(x_1, x_2, \ldots, x_n) \mid x_i \in B, 1 \leq i \leq n\}$, to B is called a **Boolean function of degree n**.

Boolean functions can be represented using expressions made up from the variables and Boolean operations.

Boolean Functions and Expressions

The **Boolean expressions** in the variables x_1, x_2, …, x_n are defined recursively as follows:

- 0, 1, x_1, x_2, …, x_n are Boolean expressions.
- If E_1 and E_2 are Boolean expressions, then $(-E_1)$, (E_1E_2), and $(E_1 + E_2)$ are Boolean expressions.

Each Boolean expression represents a Boolean function. The values of this function are obtained by substituting 0 and 1 for the variables in the expression.

Boolean Functions and Expressions

For example, we can create Boolean expression in the variables x, y, and z using the "building blocks" 0, 1, x, y, and z, and the construction rules:

Since x and y are Boolean expressions, so is xy.

Since z is a Boolean expression, so is (-z).

Since xy and (-z) are expressions, so is xy + (-z).

… and so on…

Boolean Functions and Expressions

Example: Give a Boolean expression for the Boolean function F(x, y) as defined by the following table:

x	y	F(x, y)
0	0	0
0	1	1
1	0	0
1	1	0

Possible solution: $F(x, y) = (-x) \cdot y$

Boolean Functions and Expressions

Another Example:

x	y	z	F(x, y, z)
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

Possible solution I:

$F(x, y, z) = -(xz + y)$

Possible solution II:

$F(x, y, z) = (-(xz))(-y)$

Boolean Functions and Expressions

There is a simple method for deriving a Boolean expression for a function that is defined by a table. This method is based on **minterms**.

Definition: A **literal** is a Boolean variable or its complement. A **minterm** of the Boolean variables x_1, x_2, …, x_n is a Boolean product $y_1y_2 \ldots y_n$, where $y_i = x_i$ or $y_i = -x_i$.

Hence, a minterm is a product of n literals, with one literal for each variable.

Boolean Functions and Expressions

Consider F(x,y,z) again:

x	y	z	F(x, y, z)
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

F(x, y, z) = 1 if and only if:

x = y = z = 0 or

x = y = 0, z = 1 or

x = 1, y = z = 0

Therefore,

F(x, y, z) =
(-x)(-y)(-z) +
(-x)(-y)z +
x(-y)(-z)

Boolean Functions and Expressions

Definition: The Boolean functions F and G of n variables are **equal** if and only if $F(b_1, b_2, \ldots, b_n) = G(b_1, b_2, \ldots, b_n)$ whenever $b_1, b_2, \ldots, b_n$ belong to B.

Two different Boolean expressions that represent the same function are called **equivalent**.

For example, the Boolean expressions xy, $xy + 0$, and $xy \cdot 1$ are equivalent.

Boolean Functions and Expressions

The **complement** of the Boolean function F is the function –F, where $-F(b_1, b_2, \ldots, b_n) = -(F(b_1, b_2, \ldots, b_n))$.

Let F and G be Boolean functions of degree n. The **Boolean sum F+G** and **Boolean product FG** are then defined by

$$(F + G)(b_1, b_2, \ldots, b_n) = F(b_1, b_2, \ldots, b_n) + G(b_1, b_2, \ldots, b_n)$$

$$(FG)(b_1, b_2, \ldots, b_n) = F(b_1, b_2, \ldots, b_n)\, G(b_1, b_2, \ldots, b_n)$$

Boolean Functions and Expressions

Question: How many different Boolean functions of degree 1 are there?

Solution: There are four of them, F_1, F_2, F_3, and F_4:

x	F_1	F_2	F_3	F_4
0	0	0	1	1
1	0	1	0	1

Boolean Functions and Expressions

Question: How many different Boolean functions of degree 2 are there?

Solution: There are 16 of them, F_1, F_2, …, F_{16}:

x	y	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Boolean Functions and Expressions

Question: How many different Boolean functions of degree n are there?

Solution:

There are 2^n different n-tuples of 0s and 1s.

A Boolean function is an assignment of 0 or 1 to each of these 2^n different n-tuples.

Therefore, there are 2^{2^n} different Boolean functions.

Definition of a Boolean Algebra

All the properties of Boolean functions and expressions that we have discovered also apply to **other mathematical structures** such as propositions and sets and the operations defined on them.

If we can show that a particular structure is a Boolean algebra, then we know that all results established about Boolean algebras apply to this structure.

For this purpose, we need an **abstract definition** of a Boolean algebra.

Definition of a Boolean Algebra

Definition: A Boolean algebra is a set B with two binary operations $\vee$ and $\wedge$, elements 0 and 1, and a unary operation – such that the following properties hold for all x, y, and z in B:

$x \vee 0 = x$ and $x \wedge 1 = x$ (identity laws)

$x \vee (-x) = 1$ and $x \wedge (-x) = 0$ (domination laws)

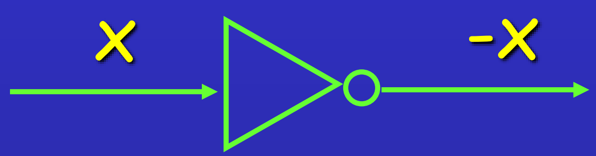
$(x \vee y) \vee z = x \vee (y \vee z)$ and
$(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and (associative laws)

$x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ (commutative laws)

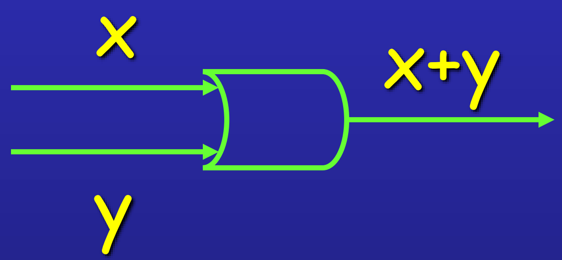
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (distributive laws)

Logic Gates

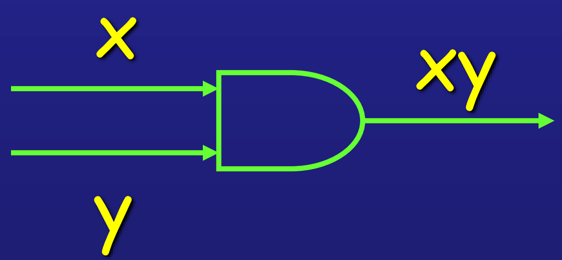
Electronic circuits consist of so-called gates.
There are three basic types of gates:



inverter



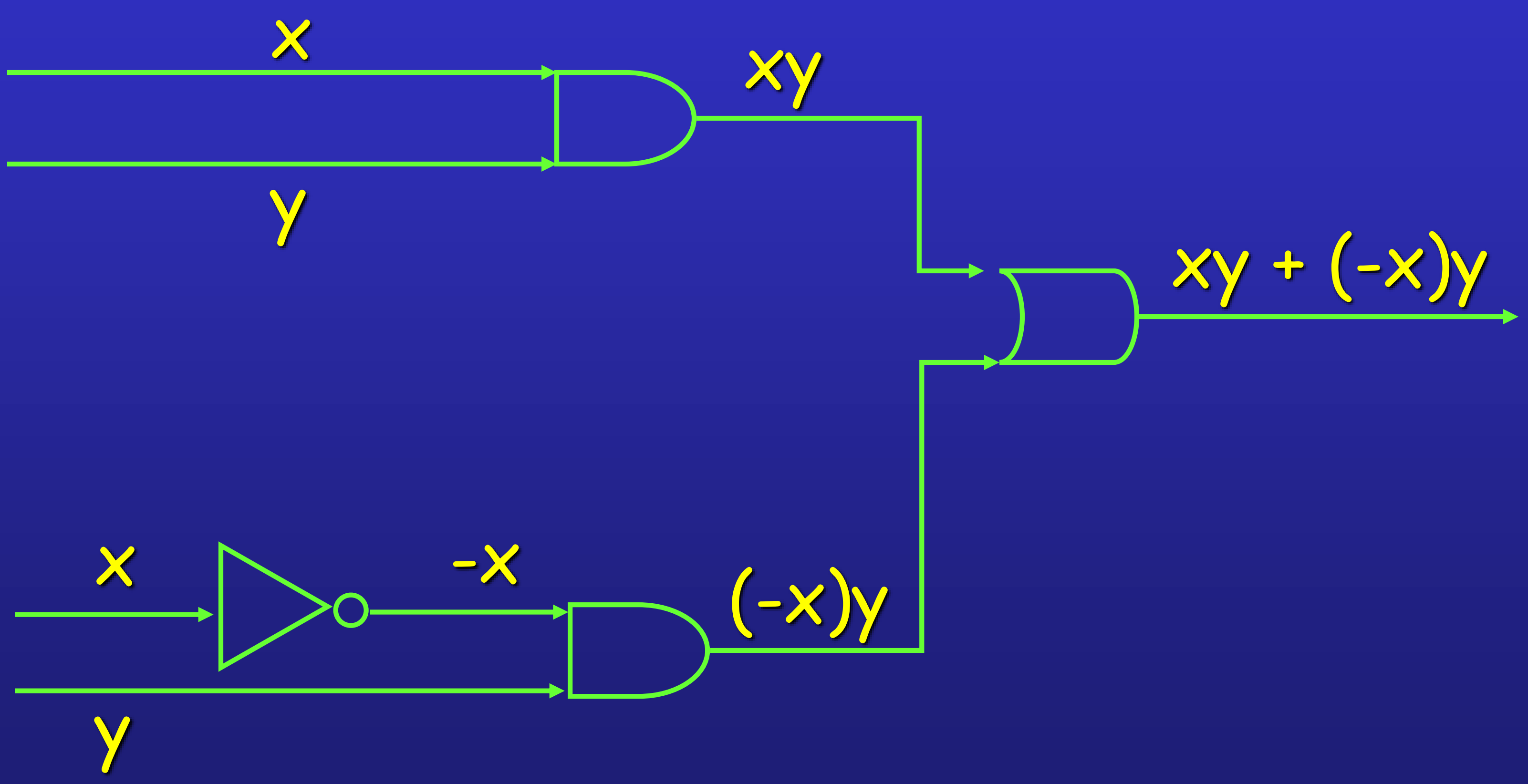
OR gate



AND gate

Logic Gates

Example: How can we build a circuit that computes the function xy + (-x)y ?



Logic, Sets, and Boolean Algebra

Logic	Set	Boolean Algebra
False	$\varnothing$	0
True	U	1
$A \wedge B$	$A \cap B$	$A \cdot B$
$A \vee B$	$A \cup B$	$A + B$
$\neg A$	A^c	$\overline{A}$

Compare the equivalence laws of them