Week # 11

- **Taylor Series**
- **Maclaurin's Series**

Angles in Degrees	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Not defined
csc	Not defined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
sec	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Not defined
cot	Not defined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

- **Maclaurin's Series:-**

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{iv}(0) + \frac{x^4}{4!} + \ldots\ldots. f^n(0)\frac{x^n}{n!}$$

- **Questions:-**

1. Find Maclaurin's Series for $f(x) = e^x$ up to four terms.

$$F(x) = e^x \longrightarrow f(0) = e^0 = 1 \longrightarrow \boxed{1}$$

$$Taking\ Higher\ Derivative$$

$$\frac{d}{dx}f(x) = \frac{d}{dx}e^x$$

$f' = e^x \longrightarrow f'(0) = e^0 = 1 \longrightarrow \boxed{2}$

$f'' = e^x \longrightarrow f''(0) = e^0 = 1 \longrightarrow \boxed{3}$

$f''' = e^x \longrightarrow f'''(0) = e^0 = 1 \longrightarrow \boxed{4}$

$f^{iv} = e^x \longrightarrow f^{iv}(0) = e^0 = 1 \longrightarrow \boxed{5}$

Using Maclaurin's Series

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{iv}(0) + \frac{x^4}{4!} + \ldots\ldots\ldots f^n(0)\frac{x^n}{n!}$$

$$f(x) = 1 + \frac{1.x}{1!} + \frac{1.x^2}{2!} + \frac{1.x^3}{3!} + \frac{1.x^4}{4!} + \cdots$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad Ans..}$$

2. Find Maclaurin's Series for $f(x) = sin\ x$ up to four terms.

$f(x) = sin\ x \longrightarrow f(0) = sin\ 0 = 0 \longrightarrow \boxed{1}$

Taking Higher Derivative

$$\frac{d}{dx}f(x) = \frac{d}{dx}sin\ x$$

$f' = cos\ x \longrightarrow f'(0) = cos0\ x = 1 \longrightarrow \boxed{2}$

$f'' = -sin\ x \longrightarrow f''(0) = -sin\ 0 = 0 \longrightarrow \boxed{3}$

$f''' = -cos\,x \longrightarrow \quad f'''(0) = -cos\,0 = -1 \longrightarrow \boxed{4}$

$f^{iv} = sin\,x \longrightarrow \quad f^{iv}(0) = sin\,0 = 0 \longrightarrow \boxed{5}$

Using Maclaurin's Series

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{iv}(0) + \frac{x^4}{4!} + \ldots\ldots.. f^n(0)\frac{x^n}{n!}$$

$$f(x) = 0 + \frac{1.x}{1!} + \frac{0.x^2}{2!} + \frac{(-1).x^3}{3!} + \frac{0.x^4}{4!} + \cdots$$

$$Sin\,x = x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + \cdots$$

$$\boxed{Sin\,x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad Ans..}$$

3. Find Maclaurin's Series for $f(x) = cos\,x$ up to four terms.

$f(x) = cos\,x \longrightarrow f(0) = cos\,0 = 1 \longrightarrow \boxed{1}$

Taking Higher Derivative

$$\frac{d}{dx}f(x) = \frac{d}{dx}cos\,x$$

$f' = -sin\,x \longrightarrow \quad f'(0) = -sin\,0 = 0 \longrightarrow \boxed{2}$

$f'' = -cos\,x \longrightarrow \quad f''(0) = -cos\,0 = -1 \longrightarrow \boxed{3}$

$f''' = sin\,x \longrightarrow \quad f'''(0) = sin\,0 = 0 \longrightarrow \boxed{4}$

$f^{iv} = cos\,x \longrightarrow \quad f^{iv}(0) = cos\,0 = 1 \longrightarrow \boxed{5}$

$$Using\ Maclaurin's\ Series$$

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{iv}(0) + \frac{x^4}{4!} + \ldots\ldots. f^n(0)\frac{x^n}{n!}$$

$$f(x) = 1 + \frac{0.x}{1!} + \frac{(-1).x^2}{2!} + \frac{(0).x^3}{3!} + \frac{1.x^4}{4!} + \cdots$$

$$cos\ x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\boxed{cos\ x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\ \ Ans..}$$

4. Find Maclaurin's Series for $f(x) = sinh\ x$ up to four terms.

$$f(x) = sinh\ x \longrightarrow f(0) = sinh\ 0 = 0 \longrightarrow \boxed{1}$$

$$Taking\ Higher\ Derivative$$

$$\frac{d}{dx}f(x) = \frac{d}{dx}\sinh x$$

$$f' = cosh\ x \longrightarrow f'(0) = cosh0\ x = 1 \longrightarrow \boxed{2}$$

$$f'' = sinh\ x \longrightarrow f''(0) = sinh\ 0 = 0 \longrightarrow \boxed{3}$$

$$f''' = cosh\ x \longrightarrow f'''(0) = cosh\ 0 = 1 \longrightarrow \boxed{4}$$

$$f^{iv} = sinh\ x \longrightarrow f^{iv}(0) = sinh\ 0 = 0 \longrightarrow \boxed{5}$$

$$Using\ Maclaurin's\ Series$$

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{iv}(0) + \frac{x^4}{4!} + \ldots\ldots. f^n(0)\frac{x^n}{n!}$$

$$f(x) = 0 + \frac{1.x}{1!} + \frac{0.x^2}{2!} + \frac{(1).x^3}{3!} + \frac{0.x^4}{4!} + \cdots$$

$$sinh\, x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\boxed{sinh\, x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \quad Ans..}$$

5. Find Maclaurin's Series for $f(x) = cosh\, x$ up to four terms.

$f(x) = cosh\, x \longrightarrow f(0) = cosh\, 0 = 1 \longrightarrow \boxed{1}$

Taking Higher Derivative

$$\frac{d}{dx} f(x) = \frac{d}{dx} \cosh x$$

$f' = sinh\, x \longrightarrow f'(0) = sinh\, 0 = 0 \longrightarrow \boxed{2}$

$f'' = cosh\, x \longrightarrow f''(0) = cosh\, 0 = 1 \longrightarrow \boxed{3}$

$f''' = sinh\, x \longrightarrow f'''(0) = sinh\, 0 = 0 \longrightarrow \boxed{4}$

$f^{iv} = cosh\, x \longrightarrow f^{iv}(0) = cosh\, 0 = 1 \longrightarrow \boxed{5}$

Using Maclaurin's Series

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{iv}(0) + \frac{x^4}{4!} + \ldots\ldots\ldots f^n(0)\frac{x^n}{n!}$$

$$cosh\,x = 1 + \frac{0.x}{1!} + \frac{(1).x^2}{2!} + \frac{(0).x^3}{3!} + \frac{1.x^4}{4!} + \cdots$$

$$cosh\,x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\boxed{cosh\,x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \quad Ans..}$$

- **Taylor Series:-**

$$f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \ldots\ldots\ldots f^n(a)\frac{(x-a)^n}{n!}$$

- **Questions:-**

1. Find Taylor Series for $f(x) = e^x \quad , \quad x = a = 1$

$$f(x) = e^x \longrightarrow f(1) = e^1 \longrightarrow \boxed{a}$$

Taking Higher Derivative

$$\frac{d}{dx}f(x) = \frac{d}{dx}e^x$$

$$f' = e^x \longrightarrow f'(1) = e^1 \longrightarrow \boxed{b}$$

$$f'' = e^x \longrightarrow f''(1) = e^1 \longrightarrow \boxed{c}$$

$$f''' = e^x \longrightarrow f'''(1) = e^1 \longrightarrow \boxed{d}$$

$$f^{iv} = e^x \longrightarrow f^{iv}(1) = e^1 \longrightarrow \boxed{e}$$

$$Using\ Taylor\ Series$$

$$f(x)=f(a)+f'(a)\frac{(x-a)}{1!}+f''(a)\frac{(x-a)^2}{2!}+f'''(a)\frac{(x-a)^3}{3!}+\ldots\ldots.f^n(a)\frac{(x-a)^n}{n!}$$

$$f(x)=f(1)+f'(1)\frac{(x-1)}{1!}+f''(1)\frac{(x-1)^2}{2!}+f'''(1)\frac{(x-1)^3}{3!}+\ldots\ldots.f^{iv}(1)\frac{(x-1)^4}{4!}+$$

$$e^x=e+e\frac{(x-1)}{1!}+e\frac{(x-1)^2}{2!}+e\frac{(x-1)^3}{3!}+e\frac{(x-1)^4}{4!}+\cdots$$

$$e^x=e\left[1+\frac{(x-1)}{1!}+\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\frac{(x-1)^4}{4!}+\cdots\right]$$

$$\boxed{e^x=e\left[1+(x-1)+\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\frac{(x-1)^4}{4!}+\cdots\right]\ Ans..}$$

2. Find Taylor Series for $f(x)=sin\,x$, $x=a=\frac{\pi}{2}$

$$f(x)=sin\,x \longrightarrow f\left(\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}\right)=1 \longrightarrow \boxed{a}$$

$$Taking\ Higher\ Derivative$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}sin\,x$$

$$f'=cos\,x \longrightarrow f'\left(\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}\right)=0 \longrightarrow \boxed{b}$$

$$f''=-sin\,x \longrightarrow f''\left(\frac{\pi}{2}\right)=-\sin\left(\frac{\pi}{2}\right)=-1 \longrightarrow \boxed{c}$$

$$f''' = -\cos x \longrightarrow f'''\left(\frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}\right) = 0 \longrightarrow \boxed{d}$$

$$f^{iv} = \sin x \longrightarrow f^{iv}\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1 \longrightarrow \boxed{e}$$

Using Taylor Series

$$f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \ldots\ldots\ldots f^n(a)\frac{(x-a)^n}{n!}$$

$$f(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\frac{(x-\frac{\pi}{2})}{1!} + f''\left(\frac{\pi}{2}\right)\frac{(x-\frac{\pi}{2})^2}{2!} + f'''\left(\frac{\pi}{2}\right)\frac{(x-\frac{\pi}{2})^3}{3!} + f^{iv}\left(\frac{\pi}{2}\right)\frac{(x-\frac{\pi}{2})^4}{4!} +$$

$$\sin x = 1 + 0\frac{(x-\frac{\pi}{2})}{1!} + (-1)\frac{(x-\frac{\pi}{2})^2}{2!} + 0\frac{(x-\frac{\pi}{2})^3}{3!} + 1\frac{(x-\frac{\pi}{2})^4}{4!} + \cdots$$

$$\sin x = 1 - 1\frac{(x-\frac{\pi}{2})^2}{2!} + 1\frac{(x-\frac{\pi}{2})^4}{4!} - 1\frac{(x-\frac{\pi}{2})^6}{6!} \ldots$$

$$\boxed{\sin x = 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} - \frac{(x-\frac{\pi}{2})^6}{6!} \ldots \quad Ans..}$$

3. Find Taylor Series for $f(x) = \cos x$, $x = a = \frac{\pi}{3}$

$$f(x) = \cos x \longrightarrow f\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \longrightarrow \boxed{a}$$

Taking Higher Derivative

$$\frac{d}{dx}f(x) = \frac{d}{dx}\cos x$$

$f' = -sin\ x \longrightarrow f'\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \longrightarrow$ b

$f'' = -cos\ x \longrightarrow f''\left(\frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2} \longrightarrow$ c

$f''' = sin\ x \longrightarrow f'''\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \longrightarrow$ d

$f^{iv} = cos\ x \longrightarrow f^{iv}\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \longrightarrow$ e

Using Taylor Series

$$f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \ldots\ldots f^n(a)\frac{(x-a)^n}{n!}$$

$$f(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\,\frac{(x-\frac{\pi}{3})}{1!} - \frac{1}{2}\,\frac{(x-\frac{\pi}{3})^2}{2!} + \frac{\sqrt{3}}{2}\,\frac{(x-\frac{\pi}{3})^3}{3!} + \frac{1}{2}\,\frac{(x-\frac{\pi}{3})^4}{4!} + \cdots$$

$$cos\ x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right) - \frac{1}{2}\,\frac{(x-\frac{\pi}{3})^2}{2\times 1} + \frac{\sqrt{3}}{2}\,\frac{\left(x-\frac{\pi}{3}\right)^3}{3\times 2\times 1} + \frac{1}{2}\,\frac{(x-\frac{\pi}{3})^4}{4\times 3\times 2\times 1} + \cdots$$

$$cos\ x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right) - \frac{1}{4}\,(x-\frac{\pi}{3})^2 + \frac{\sqrt{3}}{12}\,x-\frac{\pi}{3})^3 + \frac{1}{48}\,(x-\frac{\pi}{3})^4 + \cdots$$

$$\boxed{cos\ x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right) - \frac{1}{4}\,(x-\frac{\pi}{3})^2 + \frac{\sqrt{3}}{12}\,x-\frac{\pi}{3})^3 + \frac{1}{48}\,(x-\frac{\pi}{3})^4 + \cdots\ Ans..}$$

4. Find Taylor Series for $f(x) = tan\ x$, $x = a = \frac{\pi}{4}$

$f(x) = tanx\ x \longrightarrow f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{3}\right) = \frac{1}{2} \longrightarrow$ $\boxed{a}$

Taking Higher Derivative

$$\frac{d}{dx} f(x) = \frac{d}{dx} tan\ x$$

$f' = -sin\ x \longrightarrow f'\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \longrightarrow$ $\boxed{b}$

$f'' = -cos\ x \longrightarrow f''\left(\frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2} \longrightarrow$ $\boxed{c}$

$f''' = sin\ x \longrightarrow f'''\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \longrightarrow$ $\boxed{d}$

$f^{iv} = cos\ x \longrightarrow f^{iv}\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \longrightarrow$ $\boxed{e}$

Using Taylor Series

$$f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \ldots\ldots\ldots f^n(a)\frac{(x-a)^n}{n!}$$

$$f(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\frac{(x-\frac{\pi}{3})}{1!} - \frac{1}{2}\frac{(x-\frac{\pi}{3})^2}{2!} + \frac{\sqrt{3}}{2}\frac{(x-\frac{\pi}{3})^3}{3!} + \frac{1}{2}\frac{(x-\frac{\pi}{3})^4}{4!} + \cdots$$

$$cos\ x = \frac{1}{2} - \frac{\sqrt{3}}{2}\frac{(x-\frac{\pi}{3})}{1!} - \frac{1}{2}\frac{(x-\frac{\pi}{3})^2}{2!} + \frac{\sqrt{3}}{2}\frac{(x-\frac{\pi}{3})^3}{3!} + \frac{1}{2}\frac{(x-\frac{\pi}{3})^4}{4!} + \cdots$$

$$cos\ x = 1 - 1\frac{(x-1)^2}{2!} + 1\frac{(x-1)^4}{4!} - 1\frac{(x-1)^6}{6!} \ldots$$

$$\boxed{\sin x = 1 - \frac{(x-1)^2}{2!} + \frac{(x-1)^4}{4!} - \frac{(x-1)^6}{6!} \dots \quad Ans..}$$

5. Find Taylor Series for $f(x) = x^5 + x^4$, $x = a = 1$ up to 5 terms.

$f(x) = x^5 + x^4 \longrightarrow f(1) = (1)^5 + (1)^4 = 1 + 1 = 2 \longrightarrow \boxed{a}$

Taking Higher Derivative

$$\frac{d}{dx} f(x) = \frac{d}{dx}(x^5 + x^4)$$

$f' = 5x^4 + 4x^3 \longrightarrow f'(1) = 5(1)^4 + 4(1)^3 = 9 \longrightarrow \boxed{b}$

$f'' = 20x^3 + 12x^2 \longrightarrow f''(1) = 20(1)^3 + 12(1)^2 = 32 \longrightarrow \boxed{c}$

$f''' = 60x^2 + 24x \longrightarrow f'''(1) = 60(1)^2 + 24(1) = 84 \longrightarrow \boxed{d}$

$f^{iv} = 120x + 24 \longrightarrow f^{iv}(1) = 120(1) + 24 = 144 \longrightarrow \boxed{e}$

$f^{v} = 120 + 0 \longrightarrow f^{v}(1) = 120 + 0 = 120 \longrightarrow \boxed{f}$

Using Taylor Series

$$f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \dots\dots\dots f^n(a)\frac{(x-a)^n}{n!}$$

$$x^5 + x^4 = f(1) + f'(1)\frac{(x-1)}{1!} + f''(1)\frac{(x-1)^2}{2!} + f'''(1)\frac{(x-1)^3}{3!} + \cdots \dots.. f^{iv}(1)\frac{(x-1)^4}{4!}$$

$$+ f^{v}(1)\frac{(x-1)^5}{5!} + \cdots$$

$$x^5 + x^4 = 2 + 9\frac{(x-1)}{1!} + 32\frac{(x-1)^2}{2!} + 84\frac{(x-1)^3}{3!} + 144\frac{(x-1)^4}{4!}$$

$$+120\frac{(x-1)^5}{5!} + \cdots$$

$$x^5 + x^4 = 2 + 9(x-1) + {}^{16}\cancel{32}\frac{(x-1)^2}{\cancel{2}\times 1} + {}^{14}\cancel{84}\frac{(x-1)^3}{\cancel{3\times 2}\times 1} + {}^{6}\cancel{144}\frac{(x-1)^4}{\cancel{4\times 3}\times 2\times 1} + {}^{1}\cancel{120}\frac{(x-1)^5}{\cancel{5\times 4\times 3}\times 2\times 1}$$

$$\boxed{x^5 + x^4 = [2 + 9(x-1) + 16(x-1)^2 + 14(x-1)^3 + 6(x-1)^4 + 1(x-1)^5]\ \ Ans..}$$

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Bs C-s 1st semester

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The End of Week # 11