Week # 08

- **Chain Rule**
- **Applications Of Chain Rule**

- **Chain Rule :-**

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

- **Applications of Chain Rules:-**

1. $Find\ \frac{dy}{dx} = ?$

$x = t^2 + 3t \quad , \quad y = 16t^2$

$Sol: -$

$x = t^2 + 3t$

Differentiate w-r-t 't'

$$\frac{d}{dt}(x) = \frac{d}{dt}\ (t^2 + 3t)$$

$$\frac{dx}{dt} = \frac{d}{dt}t^2 + 3\frac{d}{dt}\ t$$

$$\frac{dx}{dt} = 2t + 3$$

$$\boxed{\frac{dt}{dx} = \frac{1}{2t+3}} \longrightarrow \text{(A)}$$

$y = 16t^2$

Differentiate w-r-t 't'

$$\frac{d}{dt}(y) = \frac{d}{dt}\ (16t^2)$$

$$\frac{dy}{dt} = 16\frac{d}{dt}\ (t^2)$$

$$\frac{dy}{dt} = 16 \times 2t$$

$$\boxed{\frac{dy}{dt} = 32t} \longrightarrow \text{(B)}$$

Now by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \longrightarrow (1)$$

Put ⟶ (A) & ⟶ (B) in ⟶ (1)

$$\frac{dy}{dx} = 32t \cdot \frac{1}{2t+3}$$

$$\boxed{\frac{dy}{dx} = \frac{32t}{2t+3} \quad Ans.}$$

2. $Find\ \frac{dy}{dx} = ?$

$x = at^3$, $y = 2at^2$

$Sol: -$

$x = at^3$

Differentiate w-r-t 't'

$$\frac{d}{dt}(x) = \frac{d}{dt}\left(at^3\right)$$

$$\frac{dx}{dt} = a\frac{d}{dt}t^3$$

$$\frac{dx}{dt} = 3at^2$$

$$\boxed{\frac{dt}{dx} = \frac{1}{3at^2}} \longrightarrow (A)$$

$y = 2at^2$

Differentiate w-r-t 't'

$$\frac{d}{dt}(y) = \frac{d}{dt}\left(2at^2\right)$$

$$\frac{dy}{dt} = 2a\frac{d}{dt}\left(t^2\right)$$

$$\frac{dy}{dt} = 2a \times 2t$$

$$\boxed{\frac{dy}{dt} = 4at} \longrightarrow (B)$$

Now by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \longrightarrow (1)$$

Put ⟶ (A) & ⟶ (B) in ⟶ (1)

$$\frac{dy}{dx} = 4at \cdot \frac{1}{3at^2}$$

$$\frac{dy}{dx} = \frac{4\cancel{a}\,\cancel{t}}{3\cancel{a}\,t^{\cancel{2}}}$$

$$\boxed{\frac{dy}{dx} = \frac{4}{3at} \quad Ans.}$$

3. $Find\ \frac{dy}{dx} = ?$

$$y = 3at^3 + 4bt^2 + c\ , \qquad x = 5at^3 + 9t^2$$

$Sol: -$

$$y = 3at^3 + 4bt^2 + c$$

Differentiate w-r-t 't'

$$\frac{d}{dt}(y) = \frac{d}{dt}\ (3at^3 + 4bt^2 + c\ )$$

$$\frac{dy}{dt} = 3a\frac{d}{dt}t^3 + 4b\frac{d}{dt}t^2 + \frac{d}{dt}c$$

$$\frac{dy}{dt} = (3a \times 3t^2) + (4b \times 2t) + 0$$

$$\frac{dy}{dt} = 9at^2 + 8bt$$

$$\boxed{\frac{dy}{dt} = 9at^2 + 8bt} \longrightarrow (A)$$

$$x = 5at^3 + 9t^2$$

Differentiate w-r-t 't'

$$\frac{d}{dt}(x) = \frac{d}{dt}\ (5at^3 + 9t^2)$$

$$\frac{dx}{dt} = 5a\frac{d}{dt}\ (t^3) + 9\frac{d}{dt}(t^2)$$

$$\frac{dx}{dt} = (5a \times 3t^2) + (9 \times 2t)$$

$$\frac{dx}{dt} = 15at^2 + 18t$$

$$\boxed{\frac{dt}{dx} = \frac{1}{15at^2+18t}} \longrightarrow (B)$$

Now by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \longrightarrow (1)$$

Put ⟶ (A) & ⟶ (B) in ⟶ (1)

$$\frac{dy}{dx} = 9at^2 + 8bt \cdot \frac{1}{15at^2 + 18t}$$

$$\frac{dy}{dx} = \frac{9at^2 + 8bt}{15at^2 + 18t}$$

$$\frac{dy}{dx} = \frac{t(9at + 8b)}{t(15at + 18)}$$

$$\frac{dy}{dx} = \frac{\cancel{t}(9at + 8b)}{\cancel{t}(15at + 18)}$$

$$\boxed{\frac{dy}{dx} = \frac{9at + 8b}{15at + 18} \quad Ans.}$$

4. $Find \ \frac{dy}{dx} = ?$

$$y = t^4 + t^3 + 3t^2 \ , \qquad x = \frac{t^3 - 1}{t}$$

$Sol: -$

$y = t^4 + t^3 + 3t^2$

Differentiate w-r-t 't'

$$\frac{d}{dt}(y) = \frac{d}{dt}\left(t^4 + t^3 + 3t^2\right)$$

$x = \frac{t^3 - 1}{t}$

Differentiate w-r-t 't'

$$\frac{d}{dt}(x) = \frac{d}{dt}\left(\frac{t^3 - 1}{t}\right)$$

$$\frac{dy}{dt} = \frac{d}{dt}t^4 + \frac{d}{dt}t^3 + 3\frac{d}{dt}t^2$$

$$\frac{dy}{dt} = 4t^3 + 3t^2 + (3 \times 2t)$$

$$\frac{dy}{dt} = 4t^3 + 3t^2 + 6t$$

$$\boxed{\frac{dy}{dt} = 4t^3 + 3t^2 + 6t} \longrightarrow \text{(A)}$$

$$\frac{dx}{dt} = \frac{t\,\frac{d}{dt}(t^3-1)-(t^3-1)\frac{d}{dt}(t)}{t^2}$$

$$\frac{dx}{dt} = \frac{t\ (3t^2-0)-(t^3-1)(1)}{t^2}$$

$$\frac{dx}{dt} = \frac{3t^3-t^3+1}{t^2}$$

$$\frac{dx}{dt} = \frac{\cancel{t^2}(3t - t + 1)}{\cancel{t^2}}$$

$$\boxed{\frac{dt}{dx} = \frac{1}{3t-t+1}} \longrightarrow \text{(B)}$$

Now by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \longrightarrow (1)$$

Put $\longrightarrow$ (A) & $\longrightarrow$ (B) in $\longrightarrow$ (1)

$$\frac{dy}{dx} = 4t^3 + 3t^2 + 6t \quad \cdot \frac{1}{3t - t + 1}$$

$$\frac{dy}{dx} = \frac{\cancel{t}(4t^2 + 3t + 6)}{\cancel{t}(3 - 1 + 1)}$$

$$\frac{dy}{dx} = \frac{4t^2 + 3t + 6}{3 - \cancel{1} + \cancel{1}}$$

$$\boxed{\frac{dy}{dx} = \frac{4t^2 + 3t + 6}{3}}$$

5. $Find\ \frac{dy}{dx} = ?$

$y = acost \ , \qquad x = btant$

$Sol: -$

$y = acost$

Differentiate w-r-t 't'

$$\frac{d}{dt}(y) = \frac{d}{dt}\ (\ acost\ )$$

$$\frac{dy}{dt} = a\frac{d}{dt}cost$$

$$\frac{dy}{dt} = a(-sint)$$

$$\frac{dy}{dt} = -asint$$

$$\boxed{\frac{dy}{dt} = -asint} \longrightarrow (A)$$

$x = btant$

Differentiate w-r-t 't'

$$\frac{d}{dt}(x) = \frac{d}{dt}\ (btant)$$

$$\frac{dx}{dt} = b\ \frac{d}{dt}tant$$

$$\frac{dx}{dt} = b(sec^2t)$$

$$\boxed{\frac{dt}{dx} = \frac{1}{bsec^2t}} \longrightarrow (B)$$

Now by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \longrightarrow (1)$$

Put $\longrightarrow$ (A) & $\longrightarrow$ (B) in $\longrightarrow$ (1)

$$\frac{dy}{dx} = -asint \cdot \frac{1}{bsec^2t}$$

$$\frac{dy}{dx} = \frac{-asint}{bsec^2t}$$

$$\frac{dy}{dx} = \frac{-asint}{b} \cdot \frac{1}{sec^2t}$$

$$\frac{dy}{dx} = \frac{-asint}{b} \cdot cos^2t$$

$$\boxed{\frac{dt}{dx} = \frac{-a}{b}\ sint\ cos^2t}$$

6. $Find\ \frac{dy}{dx} = ?$

$$y = (1 + cos^2t), \qquad x = (1 - sin^2t)$$

$Sol: -$

$$y = 1 + cos^2t$$

Differentiate w-r-t 't'

$$\frac{d}{dt}(y) = \frac{d}{dt}(1 + cos^2t)$$

$$\frac{dy}{dt} = \frac{d}{dt}1 + \frac{d}{dt}cos^2t$$

$$\frac{dy}{dt} = 0 + 2cost\frac{d}{dt}cost$$

$$\frac{dy}{dt} = 2cost(-sint)$$

$$\boxed{\frac{dy}{dt} = -2sint.cost} \longrightarrow \text{(A)}$$

$$x = 1 - sin^2t$$

Differentiate w-r-t 't'

$$\frac{d}{dt}(x) = \frac{d}{dt}(1 - sin^2t)$$

$$\frac{dx}{dt} = \frac{d}{dt}1 - \frac{d}{dt}sin^2t$$

$$\frac{dx}{dt} = 0 - 2sint\frac{d}{dt}sint$$

$$\frac{dy}{dt} = -2sint(cost)$$

$$\boxed{\frac{dt}{dx} = \frac{1}{-2sint.cost}} \longrightarrow \text{(B)}$$

Now by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \longrightarrow (1)$$

Put $\longrightarrow$ (A) & $\longrightarrow$ (B) in $\longrightarrow$ (1)

$$\frac{dy}{dx} = -2sint.cost \cdot \frac{1}{-2sint.cost}$$

$$\frac{dy}{dx} = \frac{\cancel{-2\sin t.\cos t}}{\cancel{-2\sin t.\cos t}}$$

$$\boxed{\frac{dt}{dx} = 1 \ \ Ans.}$$

7. $Find\ \frac{dy}{dx} = ?$

$y = tanh^{-1}(t)$, $x = sinh^{-1}(t)$

$Sol: -$

$y = tanh^{-1}(t)$

Differentiate w-r-t 't'

$$\frac{d}{dt}(y) = \frac{d}{dt}\left(tanh^{-1}(t)\right)$$

$$\frac{dy}{dt} = \frac{1}{1-t^2} \cdot \frac{d}{dt}(t)$$

$$\frac{dy}{dt} = \frac{1}{1-t^2} \cdot 1$$

$$\boxed{\frac{dy}{dt} = \frac{1}{1-t^2}} \longrightarrow (A)$$

$x = sinh^{-1}(t)$

Differentiate w-r-t 't'

$$\frac{d}{dt}(x) = \frac{d}{dt}\left(sinh^{-1}(t)\right)$$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1+t^2}} \cdot \frac{d}{dt}(t)$$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1+t^2}} \cdot 1$$

$$\boxed{\frac{dt}{dx} = \sqrt{1+t^2}} \longrightarrow (B)$$

Now by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \longrightarrow (1)$$

Put $\longrightarrow$ (A) & $\longrightarrow$ (B) in $\longrightarrow$ (1)

$$\frac{dy}{dx} = \frac{1}{1-t^2} \cdot \sqrt{1+t^2}$$

$$\boxed{\frac{dy}{dx} = \frac{\sqrt{1+t^2}}{1-t^2}} \; Ans.$$

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The End of Week # 08