Week # 07

- **Derivative Of trigonometric Function**
- **Derivative Of Inverse trigonometric Function**

- **Derivative Of Trigonometric Function :-**

1. $\frac{\mathrm{d}}{\mathrm{dx}}(sinx) = cosx$
2. $\frac{\mathrm{d}}{\mathrm{dx}}(cosx) = -sinx$
3. $\frac{\mathrm{d}}{\mathrm{dx}}(tanx) = sec^2x$
4. $\frac{\mathrm{d}}{\mathrm{dx}}(cotx) = -cose^2x$
5. $\frac{\mathrm{d}}{\mathrm{dx}}(secx) = secx.tanx$
6. $\frac{\mathrm{d}}{\mathrm{dx}}(cosecx) = -cosecx.cotx$

- **Proof:-**

1. Show that $\frac{\mathrm{d}}{\mathrm{dx}}(sinx) = cosx$ by using first principle Rule.

$Sol: -$

$Let\ f(x) = sinx$ ⟶ (i)

$Let\ f(x + \Delta x\ ) = sin\ x + \Delta x$ ⟶ (ii)

$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ ⟶ (iii)

Put ⟶ (i) and ⟶ (ii) in ⟶ (iii)

$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{sin\ x+\Delta x\ \ -sinx}{\Delta x}$

$\because A = x + \Delta x\ ,\ B = x$

1. $sinA - sinB = 2cos\frac{(A+B)}{2}.sin\frac{(A-B)}{2}$
2. $cosA - cosB = 2sin\frac{(A+B)}{2}.sin\frac{(A-B)}{2}$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{2\cos\left(\frac{x+\Delta x+x}{2}\right).\sin\left(\frac{x+\Delta x-x}{2}\right)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{2\cos\left(\frac{x+\Delta x+x}{2}\right).\sin\left(\frac{\cancel{x}+\Delta x-\cancel{x}}{2}\right)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{2\cos\left(\frac{2x+\Delta x}{2}\right).\sin\left(\frac{\Delta x}{2}\right)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\cos\left(\frac{2x+\Delta x}{2}\right).\sin\left(\frac{\Delta x}{2}\right)}{\Delta x/2}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\cos\left(x+\frac{\Delta x}{2}\right).\sin\left(\frac{\Delta x}{2}\right)}{\Delta x/2}$$

$$\frac{dy}{dx} = \cos\left(x+\frac{\Delta x}{2}\right).\lim_{\Delta x \to 0} \frac{\sin\left(\frac{\Delta x}{2}\right)}{\Delta x/2} \qquad \because \lim_{\Delta x \to 0} \frac{\sin\left(\frac{\Delta x}{2}\right)}{\Delta x/2} = 1$$

$$\frac{dy}{dx} = \cos\left(x+\frac{\Delta x}{2}\right).1$$

Apply the Limit

$$\frac{dy}{dx} = \cos\left(x+\frac{0}{2}\right)$$

$$\frac{dy}{dx} = \cos(x+0)$$

$$\boxed{\frac{dy}{dx} = \cos x \quad Proved.}$$

2. Show that $\frac{\mathrm{d}}{\mathrm{dx}}(cosx) = -sin\ x$ by using first principle Rule.

$Sol: -$

$Let\ f(x) = cosx$ ⟶ (i)

$Let\ f(x+\Delta x) = cos\ x + \Delta x$ ⟶ (ii)

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$ ⟶ (iii)

Put ⟶ (i) and ⟶ (ii) in ⟶ (iii)

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{cos\ x+\Delta x\ - cos\ x}{\Delta x}$$

3. $sinA - sinB = 2cos\frac{(A+B)}{2}.sin\frac{(A-B)}{2}$

4. $cosA - cosB = -2sin\frac{(A+B)}{2}.sin\frac{(A-B)}{2}$

$\because A = x + \Delta x\ ,\ B = x$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{-2\sin\left(\frac{x+\Delta x+x}{2}\right).\sin\left(\frac{x+\Delta x-x}{2}\right)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{-2\sin\left(\frac{x+\Delta x+x}{2}\right).\sin\left(\frac{\cancel{x}+\Delta x-\cancel{x}}{2}\right)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{-2\sin\left(\frac{2x+\Delta x}{2}\right).\sin\left(\frac{\Delta x}{2}\right)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{-2\sin\left(\frac{2x+\Delta x}{2}\right).\sin\left(\frac{\Delta x}{2}\right)}{\Delta x/2}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{-\sin\left(x+\frac{\Delta x}{2}\right).\sin\left(\frac{\Delta x}{2}\right)}{\Delta x/2}$$

$$\frac{dy}{dx} = -\sin\left(x+\frac{\Delta x}{2}\right).\lim_{\Delta x \to 0} \frac{\sin\left(\frac{\Delta x}{2}\right)}{\Delta x/2} \qquad \because \lim_{\Delta x \to 0} \frac{\sin\left(\frac{\Delta x}{2}\right)}{\Delta x/2} = 1$$

$$\frac{dy}{dx} = -\sin\left(x + \frac{\Delta x}{2}\right).1$$

Apply the Limit

$$\frac{dy}{dx} = -\sin\left(x + \frac{0}{2}\right)$$

$$\frac{dy}{dx} = -\sin(x + 0)$$

$$\boxed{\frac{dy}{dx} = -\sin x \ \ Proved.}$$

3. Show that $\frac{d}{dx}(tanx) = sec^2 x$ by using first principle Rule.

$Sol: -$

$Let\ f(x) = tanx$ ⟶ (i)

$Let\ f(x + \Delta x) = tan\ x + \Delta x$ ⟶ (ii)

$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ ⟶ (iii)

Put ⟶ (i) and ⟶ (ii) in ⟶ (iii)

$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{tan\ (x+\Delta x)\ - tan\ x}{\Delta x}$

$\because \alpha = x + \Delta x\ ,\ \beta = x$

5. $\sin(\alpha - \beta) = \sin\alpha\ .cos\beta - cos\beta.\sin\alpha$

$tanx = \frac{sinx}{cosx}$

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{\sin(x + \Delta x)}{\cos(x + \Delta x)} - \frac{sinx}{cosx}\right]$$

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{\sin(x+\Delta x).cosx - \cos(x+\Delta x).sinx}{\cos(x+\Delta x).cosx}\right]$$

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{\sin(\cancel{x}+\Delta x-\cancel{x}}{\cos(x+\Delta x).\cos x}\right]$$

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{\sin \Delta x}{\cos(x+\Delta x).cosx}\right]$$

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}.\lim_{\Delta x \to 0} \frac{1}{\cos(x+\Delta x).cosx} \qquad Apply\ the\ Limit$$

$$=> \quad 1\ .\ \frac{1}{\cos(x+0).cosx}$$

$$=> \quad \frac{1}{\cos x.cosx}$$

$$=> \quad \frac{1}{\cos^2 x} \qquad \because \frac{1}{\cos x} = secx$$

$$\boxed{\frac{dy}{dx} = \sec^2 x \quad Proved.}$$

4. Show that $\frac{\mathrm{d}}{\mathrm{dx}}(secx) = secx.tanx$ by using first Principle Rule.

$Sol: -$

$Let\ f(x) = secx$ ⟶ (i)

$Let\ f(x+\Delta x) = sec\ x+\Delta x$ ⟶ (ii)

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \longrightarrow \text{(iii)}$$

Put → (i) and → (ii) in → (iii)

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{sec\ (x+\Delta x)\ - sec\ x}{\Delta x}$$

$\because\ \alpha = x + \Delta x\ ,\ \beta = x$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\frac{1}{cos(x+\Delta x)} - \frac{1}{\cos x}}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{1}{cos(x+\Delta x)} - \frac{1}{\cos x}\right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{cosx - \cos\ (x+\Delta x)}{\cos x\ .cos(x+\Delta x)}\right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{cosx - \cos\ (x+\Delta x)}{\cos x\ .cos(x+\Delta x)}\right]$$

6. $sinA - sinB = 2cos\frac{(A+B)}{2}.sin\frac{(A-B)}{2}$

7. $cosA - cosB = -2sin\frac{(A+B)}{2}.sin\frac{(A-B)}{2}$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{-2sin\frac{x+(x+\Delta x)}{2}.sin\frac{x-(x+\Delta x)}{2}}{\cos x\ .cos(x+\Delta x)}\right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\frac{-2 \sin \frac{\cancel{2} x}{\cancel{2}} + \frac{\Delta x}{2} . \sin \frac{\cancel{x} - \cancel{x} - \Delta x}{2}}{\cos x . \cos(x + \Delta x)} \right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\frac{-2 \sin x + \frac{\Delta x}{2} \ . -sin \frac{\Delta x}{2}}{\cos x \ . cos(x + \Delta x)} \right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\frac{\cancel{-} 2 \sin x + \frac{\Delta x}{2} . \cancel{-} \sin \frac{\Delta x}{2}}{\cos x . \cos(x + \Delta x)} \right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\frac{2 \sin x + \frac{\Delta x}{2} \ . sin \frac{\Delta x}{2}}{\cos x \ . cos(x + \Delta x)} \right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \ sin \left[x + \frac{\Delta x}{2} \right] . \frac{1}{\cos x} . \frac{1}{\cos x + \Delta x} . \frac{sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

Apply the Limit

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \ sin \left[x + \frac{\Delta x}{2} \right] . \lim_{\Delta x \to 0} \frac{1}{\cos x} . \lim_{\Delta x \to 0} \frac{1}{\cos x + \Delta x} . \lim_{\Delta x \to 0} \frac{sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$\frac{dy}{dx} = sin \left[x + \frac{0}{2} \right] . \frac{1}{\cos x} . \frac{1}{\cos x + 0} . 1$$

$$\because \lim_{\Delta x \to 0} \frac{sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1$$

$$\frac{dy}{dx} = sin \ x . \frac{1}{\cos x} . \frac{1}{\cos x}$$

$$\frac{dy}{dx} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \qquad \because \frac{1}{\cos x} = secx \qquad \because \frac{\sin x}{\cos x} = \tan x$$

$$\boxed{\frac{dy}{dx} = \tan x . \sec x \ Proved}$$

5. Show that $\frac{d}{dx}(cosecx) = -cosecx . \cot x$ by using first Principle Rule.

$Sol: -$

$Let\ f(x) = cosec\ x \longrightarrow$ (i)

$Let\ f(x + \Delta x) = cosecx\ x + \Delta x \longrightarrow$ (ii)

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \longrightarrow \text{(iii)}$$

Put $\longrightarrow$ (i) and $\longrightarrow$ (ii) in $\longrightarrow$ (iii)

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{cosecx\ (x+\Delta x)\ - cosecx\ x}{\Delta x}$$

$\because \alpha = x + \Delta x\ ,\ \beta = x$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\frac{1}{\sin(x+\Delta x)} - \frac{1}{\sin x}}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\frac{1}{\sin(x+\Delta x)} - \frac{1}{\sin x} \right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\frac{\sin x - \sin(x+\Delta x)}{\sin x\ . \sin(x+\Delta x)} \right]$$

8. $sinA - sinB = 2cos\frac{(A+B)}{2} . sin\frac{(A-B)}{2}$

9. $cosA - cosB = -2sin\frac{(A+B)}{2} . sin\frac{(A-B)}{2}$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\frac{2cos\,\frac{x + (x + \Delta x)}{2} \,.\, sin\,\frac{x - (x + \Delta x)}{2}}{\sin x \;\; .\, sin(x + \Delta x)} \right]$$

$$\left[\frac{2\cos\frac{\cancel{2}x}{\cancel{2}} + \frac{\Delta x}{2} .\sin\frac{\cancel{x} - \cancel{x} - \Delta x}{2}}{\sin x .\sin(x + \Delta x)} \right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\frac{2\cos x + \frac{\Delta x}{2} \;.\, sin\frac{-\Delta x}{2}}{\sin x \;\; .\, sin(x + \Delta x)} \right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\frac{-2\cos x + \frac{\Delta x}{2} \;.\, sin\frac{\Delta x}{2}}{\sin x \;\; .\, sin(x + \Delta x)} \right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \; - cos\,\left[x + \frac{\Delta x}{2}\right] . \frac{1}{\sin x} . \frac{1}{\sin x + \Delta x} . \frac{sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \qquad \textit{Apply the Limit}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \; - cos\,\left[x + \frac{\Delta x}{2}\right] . \lim_{\Delta x \to 0} \frac{1}{\sin x} . \lim_{\Delta x \to 0} \frac{1}{\sin x + \Delta x} . \lim_{\Delta x \to 0} \frac{sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$\frac{dy}{dx} = -cos\left[x + \frac{0}{2}\right] . \frac{1}{\sin x} . \frac{1}{\sin x + 0} . 1 \qquad \because \lim_{\Delta x \to 0} \frac{sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1$$

$$\frac{dy}{dx} = -cos\ x\ .\frac{1}{\sin x}\ .\frac{1}{\sin x}$$

$$\frac{dy}{dx} = \frac{-cos\ x}{\sin x}\ .\frac{1}{\sin x} \qquad \because\ \frac{1}{\sin x} = cosecx \qquad \because\ \frac{cos\ x}{sin\ x} = cot\ x$$

$$\boxed{\frac{dy}{dx} = -cot\ x.cosec\ x\ Proved}$$

6. Show that $\frac{d}{dx}(cotx) = -cosec^2x$ by using first principle Rule.

$Sol: -$

$Let\ f(x) = cotx$ ⟶ (i)

$Let\ f(x + \Delta x\ ) = cot\ x + \Delta x$ ⟶ (ii)

$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ ⟶ (iii)

Put ⟶ (i) and ⟶ (ii) in ⟶ (iii)

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{cotx\ (x+\Delta x)\ -cot\ x}{\Delta x}$$

$\because\ \alpha = x + \Delta x\ ,\ \beta = x$

10. $\sin(\alpha - \beta) = \sin\alpha\ .cos\beta - cos\beta.\sin\alpha$

$cot\ x = \frac{cos\ x}{sin\ x}$

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{\cos(x+\Delta x)}{\sin(x+\Delta x)} - \frac{cos\ x}{sin\ x}\right]$$

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{sin\ x.\cos(x+\Delta x\ ).-\cos x\ .\sin(x+\Delta x)}{\sin(x+\Delta x).sin\ x}\right]$$

$$\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\frac{sin\ x-(x+\Delta x\ )}{\sin(x+\Delta x).\ sin\ x}\right]$$

$$\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\frac{\sin\ \cancel{x}-\cancel{x}-\Delta x}{\sin(x+\Delta x).\sin\ x}\right]$$ $\because\ sin-\Delta x\ =-sin\Delta x$

$$\lim_{\Delta x\to 0}\frac{-\sin\Delta x}{\Delta x}\ .\lim_{\Delta x\to 0}\frac{1}{\sin(x+\Delta x\ )\ .\ sin\ x}$$ *Apply the Limit*

$$=>\quad -1\ .\ \frac{1}{\sin(x+0\ )\ .\ sin\ x}$$

$$=>\ \frac{1}{-\sin x\ .\ sin\ x}$$

$$=>\ \frac{1}{-\sin^2 x}$$ $\because\ \frac{1}{\sin x}=cosec\ x$

$$\boxed{\frac{dy}{dx}=-\text{cosec}^2\,x\ \ Proved.}$$

- **Questions:-**

1. Use any suitable rule of differentiation to perform $\frac{dy}{dx}$ for the following functions.

a. $y = sin^3x$

$Sol:-$

$y = sin^3x$

Differentiate $w-r-t$ 'x'

b. $y = sin\ 3x + tan\ 4x$

$Sol:-$

$y = sin\ 3x + tan\ 4x$

Differentiate $w-r-t$ 'x'

$\frac{d}{dx}(y) = \frac{dy}{dx}(sin^3 x)$

Using Power Rule

$\frac{dy}{dx} = 3sin^{3-1}x \frac{dy}{dx} \text{sin}$

$\boxed{\frac{dy}{dx} = 3sin^2 . cos\ x\ Ans..}$

$\frac{d}{dx}(y) = \frac{d}{dx}(sin\ 3x + tan\ 4x)$

$\frac{dy}{dx} = \frac{d}{dx} sin\ 3x + \frac{d}{dx} tan\ 4x$

$\frac{dy}{dx} = \cos(3x) \frac{d}{dx}\ 3x + sec^2(4x) \frac{d}{dx}\ 4x$

$\frac{dy}{dx} = 3\cos(3x) + 4sec^2(4x)$

$\boxed{\frac{dy}{dx} = 3\cos\ 3x + 4sec^2 4x\ \ Ans..}$

c. $y = sin\ x. cos\ x$

$Sol: -$

$y = sin\ x. cos\ x$

Differentiate $w - r - t$ 'x'

$\frac{d}{dx}(y) = \frac{d}{dx}(sin\ x. cos\ x)$

Using Product Rule

$\frac{dy}{dx} = sin\ x \frac{d}{dx}. cos\ x + cos\ x \frac{d}{dx} sin\ x$

$\frac{dy}{dx} = sin\ x.(-sin\ x) + cos\ x(cos\ x)$

$\frac{dy}{dx} = -sin^2 x + cos^2 x$

$\boxed{\frac{dy}{dx} = -sin^2 x + cos^2 x\ \ Ans..}$

e. $y = x^2. tan\ \frac{x}{2}$

$Sol: -$

$y = x^2. tan\ \frac{x}{2}$

Differentiate $w - r - t$ 'x'

d. $y = sec\sqrt{x}$

$Sol: -$

$y = secx^{\frac{1}{2}}$

Differentiate $w - r - t$ 'x'

$\frac{d}{dx}(y) = \frac{d}{dx}(sec\ x^{\frac{1}{2}})$

Using Power Rule

$\frac{dy}{dx} = \frac{d}{dx}(sec\ x^{\frac{1}{2}})$

$\frac{dy}{dx} = sec\ \sqrt{x}. tan\sqrt{x} \frac{d}{dx}(x^{\frac{1}{2}})$

$\frac{dy}{dx} = sec\ \sqrt{x}\ .\ tan\sqrt{x}\ .\frac{1}{2}x^{\frac{1}{2}-1}$

$\frac{dy}{dx} = sec\ \sqrt{x}\ .\ tan\sqrt{x}\ .\frac{1}{2}x^{\frac{1-2}{2}}$

$\frac{dy}{dx} = sec\ \sqrt{x}\ .\ tan\sqrt{x}\ .\frac{1}{2}x^{\frac{-1}{2}}$

$\frac{dy}{dx} = sec\ \sqrt{x}\ .\ tan\sqrt{x}\ .\frac{1}{2x^{\frac{1}{2}}}$

$\boxed{\frac{dy}{dx} = \frac{sec\ \sqrt{x}\ .\ tan\sqrt{x}}{2\sqrt{x}}\ \ Ans..}$

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^2.tan\ \frac{x}{2})$$

$$\frac{dy}{dx} = x^2.\frac{d}{dx}tan\ \frac{x}{2} + tan\ \frac{x}{2}\ \frac{d}{dx}x^2)$$

$$\frac{dy}{dx} = x^2.sec^2\left(\frac{x}{2}\right)\frac{d}{dx}\left(\frac{x}{2}\right) + tan\ \frac{x}{2}\ (2x)$$

$$\frac{dy}{dx} = x^2sec^2\left(\frac{x}{2}\right).\left(\frac{1}{2}\right) + tan\ \frac{x}{2}\ (2x)$$

$$\boxed{\frac{dy}{dx} = \frac{x^2}{2}sec^2\left(\frac{x}{2}\right) + 2x.tan\ \frac{x}{2}\ \ Ans..}$$

f. $y = \frac{(cos^2 3t)}{(1+t^2)}$

$Sol: -$

$$y = \frac{(cos^2 3t)}{(1+t^2)}$$

Differentiate $w - r - t$ ‘t’

$$\frac{d}{dt}(y) = \frac{d}{dt}\left[\frac{cos^2 3t}{1+t^2}\right]$$

Using Quotient Rule

$$\frac{dy}{dt} = \left[\frac{(1+t^2)\frac{d}{dt}(cos^2 3t) - (cos^2 3t)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$\frac{dy}{dt} = \left[\frac{(1+t^2).2(cos\ 3t)\frac{d}{dt}(cos\ 3t) - (cos^2 3t)(0+2t)}{(1+t^2)^2}\right]$$

$$\frac{dy}{dt} = \left[\frac{(1+t^2).2cos\ 3t(-sin\ 3t)\frac{d}{dt}(3t) - 2t(cos^2 3t)}{(1+t^2)^2}\right]$$

$$\frac{dy}{dt} = \left[\frac{(1+t^2).2cos\ 3t(-sin\ 3t)(3) - 2t(cos^2 3t)}{(1+t^2)^2}\right]$$

Double Angel Formula $2sintcost = sin2t$

$$\frac{dy}{dt} = \left[\frac{(1+t^2).-3(2sin\,3t.cos\,3t).-2t(cos^2 3t)}{(1+t^2)^2}\right]$$

$$\frac{dy}{dt} = \left[\frac{-3(1+t^2).(sin2(\,3t)) - 2t(cos^2 3t)}{(1+t^2)^2}\right]$$

$$\frac{dy}{dt} = \left[\frac{-3(1+t^2).(sin\,6t) - 2t(cos^2 3t)}{(1+t^2)^2}\right]$$

$$\boxed{\frac{dy}{dt} = \left[\frac{-3(1+t^2).(sin\,6t) - 2t(cos^2 3t)}{(1+t^2)^2}\right] \; Ans..}$$

- **Derivative 0f Inverse trigonometric Function:-**

1. $\frac{dy}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{dy}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$
3. $\frac{dy}{dx}\tan^{-1}x = \frac{1}{1+x^2}$
4. $\frac{dy}{dx}cose\,c^{-1}\,x = \frac{-1}{|x|\sqrt{x^2-1}}$
5. $\frac{dy}{dx}secc^{-1}\,x = \frac{1}{|x|\sqrt{x^2-1}}$
6. $\frac{dy}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$

- **Proofs:-**

1. Show that $\frac{dy}{dx}\quad \sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$.

$Sol:-$

Suppose $y = sin^{-1}x$

$Siny = x$

Differentiate $w-r-t$ 'x'

$$\frac{d}{dx}(Sin\ y) = \frac{d}{dx}(x)$$

$$cos\ y\ \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{cos\ y} \longrightarrow \text{(i)}$$

$=> sin^2 y + \cos^2 y = 1$

$=> \cos^2 y = 1 - \sin^2 y$

Taking "$\sqrt{\ }$" on both sides

$$\sqrt{\cos^2 y} = \sqrt{1 - \sin^2 y}$$

$$Cosy = \sqrt{1 - sin^2 y}$$

Put in $\longrightarrow$ (i)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - sin^2 y}}$$

$\because\ siny = x$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}}$$

Formulas

1. $\sin^2 y + \cos^2 y = 1$
2. $1+\tan^2 y = \sec^2 y$
3. $1+\cot^2 y = \text{cosec}^2 y$

2. Show that $\frac{dy}{dx}$ $\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$.

$Sol:-$

Suppose $y = \cos^{-1}x$

$cosy = x$

Differentiate $w-r-t$ 'x'

$$\frac{d}{dx}(cos\ y) = \frac{d}{dx}(x)$$

$$-siny\ \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{siny} \longrightarrow \text{(i)}$$

=> $sin^2\text{y} + \cos^2\text{y} = 1$

=> $\sin^2\text{y} = 1 - \cos^2\text{y}$

Taking "$\sqrt{\ }$" on both sides

$$\sqrt{\sin^2 y} = \sqrt{1-\cos^2 y}$$

$$siny = \sqrt{1 - cos^2y}$$

Put in $\longrightarrow$ (i)

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - cos^2y}}$$

$\because\ cos\ y = x$

$$\boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}}}$$

Formulas

4. $\sin^2\text{y} + \cos^2\text{y} = 1$
5. $1+\tan^2\text{y} = \sec^2\text{y}$
6. $1+\cot^2\text{y} = \text{cosec}^2\text{y}$

3. Show that $\frac{dy}{dx}$ $\tan^{-1}x = \frac{1}{1+x^2}$.

$Sol{:}-$

Suppose $y = \tan^{-1}x$

$tan\ y = x$

Differentiate $w-r-t$ 'x'

$$\frac{d}{dx}(tan\ y) = \frac{d}{dx}(x)$$

$$sec^2y\ \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{sec^2y} \longrightarrow \text{(i)}$$

$=> 1 + tan^2y = \sec^2y$

Put in $\longrightarrow$ (i)

$$\frac{dy}{dx} = \frac{1}{1 + tan^2y}$$

$\because\ tan\ y = x$

$$\boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$$

Formulas

7. $\sin^2y + \cos^2y = 1$
8. $1+\tan^2y = \sec^2y$
9. $1+\cot^2y = \text{cosec}^2y$

4. Show that $\frac{dy}{dx}\ \text{cosec}^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$.

$Sol{:}-$

Suppose $y = \text{cosec}^{-1}x$

$cosec\ y = x$

Differentiate $w-r-t$ 'x'

$$\frac{d}{dx}(cosec\ y) = \frac{d}{dx}(x)$$

$$-cot\ y.cosec\ y\ \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{cosec\ y.cot\ y} \longrightarrow (i)$$

$=>\ 1 + cot^2\text{y} = \text{cosec}^2\text{y}$

$=>\ cot^2\text{y} = \text{cosec}^2\text{y} - 1$

Taking "$\sqrt{\ }$" on both sides

$$\sqrt{cot^2\ y} = \sqrt{cosec^2\ y - 1}$$

$$Coty = \sqrt{\text{cosec}^2\text{y} - 1}$$

Put in $\longrightarrow$(i)

$$\frac{dy}{dx} = \frac{-1}{cosec\ y.\sqrt{\text{cosec}^2\text{y} - 1}}$$

$\because\ cosec = x$

$$\boxed{\frac{dy}{dx} = \frac{-1}{|x|\sqrt{x^2 - 1}}}$$

Formulas

10. $\sin^2\text{y} + \cos^2\text{y} = 1$

11. $1+\tan^2\text{y} = \sec^2\text{y}$

12. $1+\cot^2\text{y} = \text{cosec}^2\text{y}$

5. Show that $\frac{dy}{dx}\ \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$.

$Sol: -$

Suppose $y = \sec^{-1}x$

$$sec\ y = x$$

Differentiate $w - r - t$ 'x'

$$\frac{d}{dx}(sec\ y) = \frac{d}{dx}(x)$$

$sec\ y.tan\ y\ \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{sec\ y.tan\ y}$ ⟶ (i)

$=> \ 1 + tan^2 y = \sec^2 y$

$=> \ tan^2 y = \sec^2 y - 1$

Taking "$\sqrt{\ }$" on both sides

$\sqrt{tan^2 y} = \sqrt{sec^2 y - 1}$

$tan\ y = \sqrt{\sec^2 y - 1}$

Put in ⟶ (i)

$\frac{dy}{dx} = \frac{1}{sec\ y.\sqrt{\sec^2 y - 1}}$

$\because\ sec = x$

$$\boxed{\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2 - 1}}}$$

Formulas

13. $\sin^2 y + \cos^2 y = 1$

14. $1+\tan^2 y = \sec^2 y$

15. $1+\cot^2 y = cosec^2 y$

6. Show that $\frac{dy}{dx}\ \cot^{-1} x = \frac{-1}{1+x^2}$.

$Sol: -$

Suppose $y = \ \cot^{-1} x$

$cot\ y = x$

Differentiate $w - r - t$ 'x'

$\frac{d}{dx}(cot\ y) = \frac{d}{dx}(x)$

$-cosec^2 y\ \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{-1}{cosec^2 y}$ ⟶ (i)

=> $1 + cot^2 y = \text{cosec}^2 y$

Put in ⟶ (i)

$$\frac{dy}{dx} = \frac{-1}{1 + cot^2 y}$$

$\because$ $\cot y = x$

$$\boxed{\frac{dy}{dx} = \frac{-1}{1 + x^2}}$$

Formulas

16. $\sin^2 y + \cos^2 y = 1$

17. $1 + \tan^2 y = \sec^2 y$

18. $1 + \cot^2 y = \text{cosec}^2 y$

- **Exponential Functions:-**

a^x , $a \neq 0$, $a > 1$ Is Exponential Function.

- **Examples:-**
  1. $(2)^x$
  2. $(5)^x$
  3. $(\frac{1}{2})^x$
  4. $(-3)^x$

Natural Exponential Functions:-

1. e^x
2. e^{2x}
3. e^{tanx}

- **Questions:-**

1. Find $\frac{dy}{dx}$ of Natural Exponential Functions.

i. $\frac{dy}{dx} = (e)^x$

$Sol: -$

ii. $\frac{dy}{dx} = (e)^{3x}$

$Sol: -$

iii. $\frac{dy}{dx} = (e)^{\sin x}$

$Sol: -$

$\frac{dy}{dx} = (e)^x$

$=> e^x \frac{d}{dx}(x)$

$e^x Ans.$

$\frac{dy}{dx} = (e)^{3x}$

$=> e^{3x} \frac{d}{dx}(3x)$

$3. e^{3x} Ans.$

$\frac{dy}{dx} = (e)^{\sin x}$

$=> e^{\sin x} \frac{d}{dx}(\sin x)$

$e^{\sin x}. \cos x Ans.$

2. Find $\frac{dy}{dx}$ of Common Exponential Functions.

i. $\frac{dy}{dx} = (a)^x$

$Sol: -$

$\frac{dy}{dx} = (a)^x$

$=> a^x \ln a$

$e^x\ Ans.$

ii. $\frac{dy}{dx} = (2)^x$

$Sol: -$

$\frac{dy}{dx} = (2)^x$

$=> 2^x \ln 2$

$2^x \ln 2\ Ans.$

iii. $\frac{dy}{dx} = (3)^x$

$Sol: -$

$\frac{dy}{dx} = (3)^x$

$=> 3^x \ln 3$

$3^x \ln 3\ Ans.$

iv. $\frac{dy}{dx} = (7)^{\sec x}$

$Sol: -$

$\frac{dy}{dx} = (7)^{\sec x}$

$=> 7^{\sec x} \ln 7 \frac{d}{dx} \sec x$

$7^{\sec x} \ln 7. \sec x. \tan x\ Ans.$

v. $\frac{dy}{dx} = (7)^{3x}$

$Sol: -$

$\frac{dy}{dx} = (7)^{3x}$

$=> 7^{3x} \ln 7 \frac{d}{dx}(3x)$

$7^{3x} \ln 7.3 Ans.$

- **Derivative Of Hyperbolic Function :-**

1. $\frac{d}{dx}(sin\ hx) = cos\ hx = \frac{e^x - e^{-x}}{2}$

2. $\frac{d}{dx}(cos\ hx) = sin\ hx = \frac{e^x + e^{-x}}{2}$

3. $\frac{d}{dx}(tan\ hx) = sech^2 x = \frac{sinh\ x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

4. $\frac{d}{dx}(cot\ hx) = -coseh^2 x = \frac{\cosh x}{sinh\ x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

5. $\frac{d}{dx}(sec\ hx) = -tan\ hx.sec\ hx = \frac{1}{cosh\ x} = \frac{2}{e^x + e^{-x}}$

6. $\frac{d}{dx}(cosec\ hx) = -cot\ hx.cosec\ hx = \frac{1}{sinh\ x} = \frac{2}{e^x - e^{-x}}$

- **Derivative Of Inverse Hyperbolic Function :-**

1. $\frac{dy}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$

2. $\frac{dy}{dx}\cosh^{-1} x = \frac{-1}{\sqrt{x^2-1}}$

3. $\frac{dy}{dx}\tanh^{-1} x = \frac{1}{1-x^2}$

4. $\frac{dy}{dx} cose\,ch^{-1}\, x = \frac{-1}{|x|\sqrt{x^2+1}}$

5. $\frac{dy}{dx} secch^{-1}\, x = \frac{-1}{|x|\sqrt{1-x^2}}$

6. $\frac{dy}{dx}\coth^{-1} x = \frac{-1}{x^2-1}$

- **Proofs:-**

1. Show that $\frac{dy}{dx} \quad \sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}$.

$Sol: -$

Suppose $y = sinh^{-1}x$

$Sinhy = x$

Differentiate $w - r - t$ 'x'

$\frac{d}{dx}(Sinh\, y) = \frac{d}{dx}(x)$

$cosh\, y \ \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{cosh\, y} \longrightarrow$ (i)

$=> \cosh^2 y - sinh^2 y = 1$

$=> \cosh^2 y = 1 + \sinh^2 y$

Taking "$\sqrt{\ }$" on both sides

$\sqrt{\cosh^2 y} = \sqrt{1 + \sinh^2 y}$

$Cosh\, y = \sqrt{1 + sinh^2 y}$

Put in $\longrightarrow$ (i)

$\frac{dy}{dx} = \frac{1}{\sqrt{1+sinh^2 y}}$

$\because \ sinhy = x$

2. Show that $\frac{dy}{dx} \quad \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$

$Sol: -$

Suppose $y = \cosh^{-1} x$

$coshy = x$

Differentiate $w - r - t$ 'x'

$\frac{d}{dx}(cosh\, y) = \frac{d}{dx}(x)$

$sinhy \ \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{sinh\, y} \longrightarrow$ (i)

$=> \cosh^2 y - sinh^2 y = 1$

$=> \cosh^2 y - 1 = sinh^2 y$

Taking "$\sqrt{\ }$" on both sides

$\sqrt{\cosh^2 y - 1} = \sqrt{\sinh^2}$

$sinh\, y = \sqrt{cosh^2 y - 1}$

Put in $\longrightarrow$ (i)

$\frac{dy}{dx} = \frac{1}{\sqrt{cosh^2 y - 1}}$

$\because \ cosh\, y = x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$$

- **Questions:-**

1. Use any suitable rule of differentiation to perform $\frac{dy}{dx}$ for the following functions.

a. $y = a^x . \sin x$

$Sol: -$

$y = a^x . \sin x$

Differentiate $w - r - t$ 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}(a^x . \sin x)$$

Using Product Rule

$$\frac{dy}{dx} = a^x . \frac{d}{dx} \sin x + \sin x \frac{d}{dx} a^x$$

$$\frac{dy}{dx} = a^x . \cos x + \sin x . a^x \ln a$$

$$\boxed{\frac{dy}{dx} = a^x . \cos x + \sin x . a^x \ln a \ Ans..}$$

b. $y = e^{ax} . \cosh x$

$Sol: -$

$y = e^{ax} . \cosh x$

Differentiate $w - r - t$ 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}(e^{ax} . \cosh x)$$

Using Product Rule

$$\frac{dy}{dx} = e^{ax} . \frac{d}{dx} \cosh x + \cosh x \frac{d}{dx} e^{ax}$$

$$\frac{dy}{dx} = e^{ax} . \sinh + \cosh x . e^{ax} \frac{d}{dx}(ax)$$

$$\boxed{\frac{dy}{dx} = e^{ax} . \sinh + a . e^{ax} \cosh x \ Ans..}$$

c. $y = \sin^{-1}(\tanh x)$

$Sol: -$

$y = \sin^{-1}(\tanh x)$

Differentiate $w - r - t$ 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}(\sin^{-1}(\tanh x))$$

$$\because \frac{dy}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

d. $y = e^{ax} . \sin^2 x$

$Sol: -$

$y = e^{ax} . \sin^2 x$

Differentiate $w - r - t$ 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}(e^{ax} . \sin^2 x)$$

Using Product Rule

$$\frac{dy}{dx}y = \frac{1}{\sqrt{1-tanh^2x}}\frac{d}{dx}(tanh\ x)$$

$$\because \frac{d}{dx}(tan\ hx) = sech^2x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-tanh^2x}}\ sech^2x$$

$$\frac{dy}{dx} = \frac{sech^2x}{\sqrt{1-tanh^2x}}$$

$$\because 1 - tanh^2x = sech^2x$$

$$\frac{dy}{dx} = \frac{\sec h^2 x}{\sqrt{\sec h^2 x}}$$

$$\frac{dy}{dx} = \frac{\sec h^2 x}{\sec hx}$$

$$\boxed{\frac{dy}{dx} = \text{sech}\,x\ Ans..}$$

$$\frac{dy}{dx} = e^{ax}.\frac{d}{dx}sin^2x + sin^2x\frac{dy}{dx}e^{ax}$$

$$\frac{dy}{dx} = e^{ax}.2\,sinx\frac{d}{dx}sinx + sin^2x.e^{ax}\frac{d}{dx}(ax)$$

$$\frac{dy}{dx} = e^{ax}.2\,sinx(cos\ x) + sin^2x.e^{ax}.a$$

$$\boxed{\frac{dy}{dx} = 2sin\ xcos\ x.e^{ax} + a.e^{ax}.sin^2x\ Ans.}$$

e. $y = \frac{\sqrt{sin\ x}}{sin\ \sqrt{x}}$

$Sol: -$

$$y = \frac{\sqrt{sin\ x}}{sin\ \sqrt{x}}$$

Differentiate $w - r - t$ 'x'

$$\frac{d}{dx}y = \frac{d}{dx}\left[\frac{\sqrt{sin\ x}}{sin\ \sqrt{x}}\right]$$

Using Quotient Rule

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$$\frac{dy}{dx} = \frac{sin\,\sqrt{x}\ .\ \frac{d}{dx}\left(\sqrt{sin\,x}\right) - \sqrt{sin\,x}\ .\ \frac{d}{dx}(sin\,\sqrt{x}\,)}{(sin\,\sqrt{x})^2}$$

$$\frac{dy}{dx} = \frac{sin\,\sqrt{x}\ .\ \frac{d}{dx}(sin\,x)^{\frac{1}{2}} - \sqrt{sin\,x}\,.\ \left(cos\,\sqrt{x}\,\right)\ .\ \frac{d}{dx}(\sqrt{x}\,)}{(sin\,\sqrt{x})^2}$$

$$\frac{dy}{dx} = \frac{sin\,\sqrt{x}\ .\ \frac{1}{2}(\sin x)^{\frac{1}{2}-1}\frac{d}{dx}(sin\,x) - \sqrt{sin\,x}\ .\ \left(cos\,\sqrt{x}\,\right)\ .\ \frac{dy}{dx}(x^{\frac{1}{2}})}{(sin\,\sqrt{x})^2}$$

$$\frac{dy}{dx} = \frac{sin\,\sqrt{x}\ .\ \frac{1}{2}(\sin x)^{\frac{1-2}{2}}(cos\,x) - \sqrt{sin\,x}\ .\ \left(cos\,\sqrt{x}\,\right)\ .\ \frac{1}{2}(x)^{\frac{1}{2}-1}}{(sin\,\sqrt{x})^2}$$

$$\frac{dy}{dx} = \frac{sin\,\sqrt{x}\ .\ \frac{1}{2}(\sin x)^{\frac{-1}{2}}(cos\,x) - \sqrt{sin\,x}\ .\ \left(cos\,\sqrt{x}\,\right)\ .\ \frac{1}{2}(x)^{\frac{-1}{2}}}{(sin\,\sqrt{x})^2}$$

$$\frac{dy}{dx} = \frac{sin\,\sqrt{x}\ .\ \frac{1}{2\sqrt{sin\,x}}(cos\,x) - \sqrt{sin\,x}\ .\ \left(cos\,\sqrt{x}\,\right)\ .\ \frac{1}{2\sqrt{x}}}{(sin\,\sqrt{x})^2}$$

The End of Week # 07