0Week # 04

- **Limits of different functions**
- **Left hand Limits and Right Hand Limits**

• Limits of different functions :-

If $f(x)$ be the any function. The limit of the function is defined as:

$$\lim_{x\to c} f(x) = L \longrightarrow \text{Unique Defined}$$

• Properties of Limit :-

1. **Sum Rule :-**

$$\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

2. **Difference Rule :-**

$$\lim_{x\to c}[f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$$

3. **Product Rule :-**

$$\lim_{x\to c}[f(x) \,.\, g(x)] = \lim_{x\to c} f(x) \,.\, \lim_{x\to c} g(x)$$

4. **Quotient Rule :-**

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to c}(x)}{\lim\limits_{x\to c}g(x)}$$

5. **Power Rule :-**

$$\lim_{x\to c}[f(x)]^n = [\lim_{x\to c}f(x)]^n$$

6. **Constant Rule :-**

$$\lim_{x\to c} = a = a$$

• Questions :-

1. $\lim\limits_{x\to 2} 4 = 4$

$Sol:-$

$\lim\limits_{x\to 2} 4 = 4$

2. $\lim\limits_{x\to 3} 3$

$Sol:-$

$\lim\limits_{x\to 3} 3$

$\boxed{4 = 4\ Ans.}$

3. $\lim\limits_{x\to c}\ x^2$

$Sol :-$

$\lim\limits_{x\to c}\ x^2$

Apply Limit

$\boxed{c^2\ Ans.}$

5. $\lim\limits_{x\to -2}\sqrt{4x^2-3}$

$Sol :-$

$\lim\limits_{x\to -2}\sqrt{4x^2-3}$

$\lim\limits_{x\to -2}(4x^2-3)^{\frac{1}{2}}$

$[\lim\limits_{x\to -2}(4x^2-3)]^{\frac{1}{2}}$

Apply Limit

$=> [\lim\limits_{x\to -2}4x^2 - \lim\limits_{x\to -2} - 3]^{\frac{1}{2}}$

$[4\,(-2)^2-3]^{\frac{1}{2}}$

$[16-3]^{\frac{1}{2}}$

$[13]^{\frac{1}{2}}$

$\boxed{\sqrt{13}\ Ans.}$

7. $\lim\limits_{t\to 6} 8(t-5)(t-7)$

$Sol :-$

$\lim\limits_{t\to 6} 8(t-5)(t-7)$

$\boxed{x = 3\ Ans.}$

4. $\lim\limits_{x\to c} x^2+5$

$Sol :-$

$\lim\limits_{x\to c} x^2+5$

Apply Limit

$=> \lim\limits_{x\to c} x^2 + \lim\limits_{x\to c}5$

c^2+5

$\boxed{c^2+5\ Ans.}$

6. $\lim\limits_{x\to 2}(-x^2+5x-2)$

$Sol :-$

$\lim\limits_{x\to 2}(\ x^2+5x-2)$

Apply Limit

$=> \lim\limits_{x\to 2}(-x^2) + \lim\limits_{x\to 2}5x - \lim\limits_{x\to 2}2$

$= -(2)^2+5(2)-2$

$= -4+10-2$

$= -4+8$

$\boxed{4\ \ Ans.}$

8. $\lim\limits_{x\to 2}\frac{x+3}{x+6}$

Sol :-

. $\lim\limits_{x\to 2}\frac{x+3}{x+6}$

Apply the Limit

$$=> \; 8\lim_{t\to 6}(t-5).\lim_{t\to 6}(t-7)$$

$$= 8\,(6-5).(6-7)$$

$$= 8(1)\,.(-1)$$

$$\boxed{-8\ Ans.}$$

1. $\lim\limits_{y\to 2} \frac{y+2}{y^2+5y+6}$

$Sol:-$

$$\lim_{y\to 2} \frac{y+2}{y^2+5y+6}$$

Apply the Limit

$$=> \frac{\lim_{y\to 2} y + \lim_{y\to 2} 2}{\lim_{y\to 2} y^2 + \lim_{y\to 2} 5y + \lim_{y\to 2} 6}$$

$$= \frac{2+2}{(2)^2+5(2)+6}$$

$$= \frac{4}{4+10+6}$$

$$= \frac{\not{4}}{\not{2}0}$$

$$\boxed{\frac{1}{5}\ Ans.}$$

11. $\lim\limits_{y\to -3} (5-y)^{\frac{4}{3}}$

$Sol:-$

Apply the Limit

$$=> \frac{\lim_{x\to 2} x+3}{\lim_{x\to 2} x+}$$

$$= \frac{2+3}{2+6}$$

$$\boxed{\frac{5}{8} Ans.}$$

10. $\lim_{x\to 1} 3(2x-1)^2$

$Sol:-$

$$\lim_{x\to 1} 3(2x-1)^2$$

Apply the Limit

$$=> 3(\lim_{x\to 1} 2x - \lim_{x\to 1} 1)^2$$

$$= 3(2(-1)-1)^2$$

$$= 3(-2-1)^2$$

$$= 3(-3)^2$$

$$= 3(9)$$

$$\boxed{27\ Ans.}$$

12. $\lim\limits_{x\to 5} \frac{x-5}{x^2-25}$

$Sol:-$

$$\lim_{y\to -3}(5-y)^{\frac{4}{3}}$$

Apply the Limit

$$=> [\lim_{y\to -3} 5 - \lim_{y\to -3} y]^{\frac{4}{3}}$$

$$= [5-(-3)]^{\frac{4}{3}}$$

$$= (5+3)^{\frac{4}{3}}$$

$$= (8)^{\frac{4}{3}}$$

$$= ((2)^{\cancel{3}})^{\frac{4}{\cancel{3}}}$$

$$= (2)^4$$

$\boxed{16\ Ans.}$

13. $$\lim_{x\to -3}\frac{x+3}{x^2+4x+3}$$

Sol: −

$$\lim_{x\to -3}\frac{x+3}{x^2+4x+3}$$

Apply the Limit

$$= \frac{-3+3}{(-3)^2+4(-3)+3}$$

$$= \frac{\cancel{0}}{\cancel{9}+\cancel{3}-1\cancel{2}} => \frac{0}{0}$$

It is $\frac{0}{0}$ Form then by Factorization

$$\lim_{x\to 5}\frac{x-5}{x^2-25}$$

Apply the Limit

$$= \frac{5-5}{5^2-25} => \frac{\cancel{5}-\cancel{5}}{\cancel{25}-\cancel{25}} => \frac{0}{0}$$

It is $\frac{0}{0}$ Form then by formula

$$a^2-b^2 = (a-b)(a+b)$$

$$= \lim_{x\to 5}\frac{x-5}{x^2-5^2}$$

$$= \lim_{x\to 5}\frac{\cancel{(x-5)}}{\cancel{(x-5)}(x+5)}$$

$$= \lim_{x\to 5}\frac{1}{(x+5)}$$

Apply the Limit

$$=> \frac{\lim_{x\to 5} 1}{\lim_{x\to 5}(x+5)}$$

$$= \frac{1}{(5+5)}$$

$\boxed{\frac{1}{10}\ Ans.}$

14. $$\lim_{x\to -5}\frac{x^2+3x-10}{x+5}$$

Sol: −

$$\lim_{x\to -5}\frac{x^2+3x-10}{x+5}$$

Apply the Limit

$$= \frac{(-5)^2+3(-5)-10}{(-5)+5} => \frac{2\cancel{5}-1\cancel{5}-1\cancel{0}}{-\cancel{5}+\cancel{5}} => \frac{0}{0}$$

$$= \lim_{x\to-3} \frac{x+3}{x^2+3x+x+3}$$

$$= \lim_{x\to-3} \frac{x+3}{x(x+3)+1(x+3)}$$

$$= \lim_{x\to-3} \frac{\cancel{(x+3)}}{\cancel{(x+3)}\,(x+1)}$$

$$= \lim_{x\to-3} \frac{1}{(x+1)}$$

Apply the Limit

$$=> \frac{\lim_{x\to-3} 1}{\lim_{x\to-3}(x+1)}$$

$$= \frac{1}{(-3+1)}$$

$$\boxed{\frac{1}{-2}\ Ans.}$$

It is $\frac{0}{0}$ Form then by Factorization

$$= \lim_{x\to-5} \frac{x^2+5x-2x-10}{x+5}$$

$$= \lim_{x\to-5} \frac{x(x+5)-2(x+5)}{x+5}$$

$$= \lim_{x\to-5} \frac{\cancel{(x+5)}\,(x-2)}{\cancel{(x+5)}}$$

$$=> \lim_{x\to-5}(x-2)$$

Apply the Limit

$$= -5-2$$

$$\boxed{-7\ Ans.}$$

15. $\lim_{x\to2} \frac{x^2-7x+10}{x-2}$

$Sol:-$

$$\lim_{x\to2} \frac{x^2-7x+10}{x-2}$$

Apply the Limit

$$\frac{(2)^2+7(2)+10}{(2)-2} => \frac{\cancel{4}-1\cancel{4}+\cancel{10}}{\cancel{2}-\cancel{2}} => \frac{0}{0}$$

It is $\frac{0}{0}$ Form then by Factorization

$$= \lim_{x\to2} \frac{x^2-2x-5x+10}{x-2}$$

$$= \lim_{x\to2} \frac{x(x-2)-5(x-2)}{x-2}$$

16. $\lim_{t\to1} \frac{t^2+t-2}{t^2-1}$

$Sol:-$

$$\lim_{t\to1} \frac{t^2+t-2}{t^2-1}$$

Apply the Limit

$$= \frac{(1)^2+1-2}{(1)^2-1} => \frac{\cancel{1}+\cancel{1}-\cancel{2}}{\cancel{1}-\cancel{1}} => \frac{0}{0}$$

It is $\frac{0}{0}$ Form then by Factorization

$$= \lim_{t\to1} \frac{t^2+2t-t-2}{t^2-1^2}$$

$$= \lim_{t\to1} \frac{t(t+2)-1(t+2)}{(t+1)(t-1)}$$

$$= \lim_{x\to 2} \frac{(x-2)(x-5)}{(x-2)}$$

$$= \lim_{x\to 2} (x-5)$$

Apply the Limit

$$=> (\lim_{x\to 2} x - \lim_{x\to 2} 5)$$

$$= (2-5)$$

$$\boxed{-3\ Ans.}$$

17. $\lim_{t\to -1} \frac{t^2+3t+2}{t^2-t-2}$

$Sol:-$

$$\lim_{t\to -1} \frac{t^2+3t+2}{t^2-t-2}$$

Apply the Limit

$$= \frac{(-1)^2+3-(1)+2}{(-1)^2-(-1)-2} => \frac{1+2-3}{1+1-2} => \frac{0}{0}$$

It is $\frac{0}{0}$ Form then by Factorization

$$= \lim_{t\to -1} \frac{t^2+t+2t+2}{t^2-2t+t-2}$$

$$= \lim_{t\to -1} \frac{t(t+1)+2(t+1)}{t(t-2)+1(t-2)}$$

$$= \lim_{t\to -1} \frac{(t+1)(t+2)}{(t-2)(t+1)}$$

$$= \lim_{t\to -1} \frac{(t+2)}{(t-2)}$$

Apply the Limit

$$= \lim_{t\to 1} \frac{(t+2)(t-1)}{(t+1)(t-1)}$$

$$= \lim_{t\to 1} \frac{(t+2)}{(t+1)}$$

Apply the Limit

$$=> \frac{\lim_{t\to 1}(t+2)}{\lim_{t\to 1}(t+1)}$$

$$= \frac{1+2}{1+1}$$

$$\boxed{\frac{3}{2}\ Ans.}$$

18. $\lim_{u\to 1} \frac{u^4-1}{u^3-1}$

Sol:-

$$\lim_{u\to 1} \frac{u^4-1}{u^3-1}$$

Apply the Limit

$$= \frac{(1)^4-1}{(1)^3-1} = \frac{1-1}{1-1} => \frac{0}{0}$$

It is $\frac{0}{0}$ Form then by Formulas

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$= \lim_{u\to 1} \frac{(u^2)^2-(1)^2}{(u)^3-(1)^3}$$

$$= \lim_{u\to 1} \frac{(u^2-1)(u^2+1)}{(u-1)(u^2+u(1))+(1)^2}$$

$$= \lim_{u\to 1} \frac{(u)^2-(1)^2\ (u^2+1)}{(u-1)(u^2+u(1))+(1)^2}$$

$$=> \frac{\lim_{t\to -1}(t+2)}{\lim_{t\to -1}(t-2)}$$

$$= \frac{(-1+2)}{(-1-2)}$$

$$\boxed{\frac{1}{-3} \ or -\frac{1}{3} Ans.}$$

19. $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$

$Sol: -$

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$$

Apply the Limit

$$= \frac{\sqrt{9}-3}{9-9} => \frac{\cancel{3}-\cancel{3}}{\cancel{9}-\cancel{9}} => \frac{0}{0}$$

It is $\frac{0}{0}$ Form then by Rationalization

$$= \lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} \times \frac{\sqrt{x}+3}{\sqrt{x}+3}$$

$$= \lim_{x\to 9} \frac{(\sqrt{x})^{\cancel{2}} - (3)^2}{(x-9)(\sqrt{x}+3)}$$

$$= \lim_{x\to 9} \frac{\cancel{(x-9)}}{\cancel{(x-9)}(\sqrt{x}+3)}$$

$$= \lim_{x\to 9} \frac{1}{\sqrt{x}+3}$$

Now Apply the

$$=> \frac{\lim_{x\to 9} 1}{\lim_{x\to 9} \sqrt{x}+3}$$

$$= \frac{1}{\sqrt{9}+3} => \frac{1}{3+3}$$

$$= \lim_{u\to 1} \frac{\cancel{(u-1)}(u+1)(u^2+1)}{\cancel{()u-1}(u^2+u+1)}$$

$$= \lim_{u\to 1} \frac{(u+1)(u^2+1)}{(u^2+u+1)}$$

Apply the Limit

$$= \frac{\lim_{u\to 1}[(u+1)(u^2+1)]}{\lim_{u\to 1} u^2 + \lim_{u\to 1} u + \lim_{u\to 1} 1)}$$

$$= \frac{(1+1)((1)^2+1)}{(1)^2+1+1}$$

$$= \frac{(2)(1+1)}{1+1+1} => \frac{(2)(2)}{3}$$

$$\boxed{\frac{4}{3} \ Ans.}$$

20. $\lim_{v\to 2} \frac{v^3-8}{v^4-16}$

$Sol: -$

$$\lim_{v\to 2} \frac{v^3-8}{v^4-16}$$

Apply the Limit

$$= \frac{2^3-8}{2^4-16} => \frac{\cancel{8}-\cancel{8}}{\cancel{16}-\cancel{16}} => \frac{0}{0}$$

It is $\frac{0}{0}$ Form then by Formulas

$$a^3 - b^3 = (a-b)(a^2+ab+b^2$$

$\boxed{\frac{1}{6}\ Ans.}$

$$a^2 - b^2 = (a+b)(a-b)$$

$$= \lim_{v\to 2} \frac{v^3-(2)^3}{(v^2)^2-(4)^2}$$

$$= \lim_{v\to 2} \frac{(v-2)(v^2+2(v)+(2)^2}{(v^2-4)(v^2+4)}$$

$$= \lim_{v\to 2} \frac{(v-2)(v^2+2(v)+(2)^2}{(v^2-4)(v^2+4)}$$

$$= \lim_{v\to 2} \frac{(v-2)(v^2+2v+4)}{(v)^2-(2)^2(v^2+4)}$$

$$= \lim_{v\to 2} \frac{\cancel{(v-2)}(v^2+2v+4)}{\cancel{(v-2)}(v+2)(v^2+4)}$$

$$= \lim_{v\to 2} \frac{(v^2+2v+4)}{(v+2)(v^2+4)}$$

Apply the Limit

$$=> \frac{\lim_{v\to 2} (v^2+2v+4)}{\lim_{v\to 2} [(v+2)(v^2+4)]}$$

$$= \frac{(2)^2+2(2)+4}{(2+2)\ ((2)^2+4)} => \frac{4+4+4}{(4)\ (4+4)}$$

$$= \frac{\cancel{12}^{1}}{\cancel{24}^{2}}$$

$\boxed{\frac{1}{2}\ Ans.}$

- **Limits Of Infinity :-**

$$\lim_{x\to\infty} \quad \text{Or} \quad \lim_{x\to -\infty}$$

- **Questions :-**

1. $\lim\limits_{x\to\infty} \dfrac{5x+9}{11x-3}$

$Sol:-$

$\lim\limits_{x\to\infty} \frac{5x+9}{11x-3}$

Apply the Limit

$= \frac{5(\infty)+9}{11(\infty)-3} \;=>\; \frac{\infty+9}{\infty-3} \;=>\; \frac{\infty}{\infty} \qquad \because\; 5(\infty)=\infty$

Hence it is $\frac{\infty}{\infty}$ form so $\times$ ing & $\div$ing

Numerator and Denominator by"x"

$$= \lim_{x\to\infty} \frac{\frac{5x+9}{x}}{\frac{11x\;-3}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{5\cancel{x}}{\cancel{x}}+\frac{9}{x}}{\frac{11\cancel{x}}{\cancel{x}}-\frac{3}{x}}$$

$$= \lim_{x\to\infty} \frac{5+\frac{9}{x}}{11-\frac{3}{x}}$$

Now Apply the Limit

$$=> \frac{\lim\limits_{x\to\infty} 5+\frac{9}{x}}{\lim\limits_{x\to\infty} 11-\frac{3}{x}}$$

$$= \frac{5+\frac{9}{\infty}}{11-\frac{3}{\infty}} \qquad \because\; \frac{9}{\infty}=0$$

$$= \frac{5+0}{11-0}$$

$\boxed{\frac{5}{11}\;\; Ans.}$

2. $\lim\limits_{x\to\infty} \dfrac{3x^2-4x+5}{9x^3+2x^2-6x+11}$

$Sol:-$

$$\lim_{x\to\infty} \frac{3x^2-4x+5}{9x^3+2x^2-6x+11}$$

Apply the Limit

$$= \frac{3(\infty)^2-4(\infty)+5}{9(\infty)^3+2(\infty)^2-6(\infty)+11} \Rightarrow \frac{\infty+5}{\infty+11} \Rightarrow \frac{\infty}{\infty}$$

Hence it is $\frac{\infty}{\infty}$ form so × ing & ÷ing

Numerator and Denominator by "x^3"

$$= \lim_{x\to\infty} \frac{\dfrac{3x^2-4x+5}{x^3}}{\dfrac{9x^3+2x^2-6x+11}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\dfrac{3\cancel{x^2}}{x^{\cancel{3}}} - \dfrac{4\cancel{x}}{x^{\cancel{3}}} + \dfrac{1}{x^3}}{\dfrac{9\cancel{x^3}}{\cancel{x^3}} + \dfrac{2\cancel{x^2}}{x^{\cancel{3}}} - \dfrac{6\cancel{x}}{x^{\cancel{3}}} + \dfrac{11}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\dfrac{3}{x} - \dfrac{4}{x^2} + \dfrac{5}{x^3}}{9 + \dfrac{2}{x} - \dfrac{6}{x^2} + \dfrac{11}{x^3}}$$

Now apply the Limit

$$\Rightarrow \frac{\lim\limits_{x\to\infty}\left[\dfrac{3}{x} - \dfrac{4}{x^2} + \dfrac{5}{x^3}\right]}{\lim\limits_{x\to\infty}\left[9 + \dfrac{2}{x} - \dfrac{6}{x^2} + \dfrac{11}{x^3}\right]}$$

$$= \frac{\frac{3}{\infty} - \frac{4}{\infty^2} + \frac{5}{\infty^3}}{9 + \frac{2}{\infty} - \frac{6}{\infty^2} + \frac{11}{\infty^3}}$$

$$= \frac{0 - 0 + 0}{9 + 0 - 0 + 0} \Rightarrow \frac{0}{9}$$

$$\boxed{0 \ \ Ans.}$$

3. $\lim\limits_{x\to\infty} \dfrac{3x^2 - 5x + 1}{5x^2 - 3x + 1}$

$Sol :-$

$$\lim_{x\to\infty} \frac{3x^2 - 5x + 1}{5x^2 - 3x + 1}$$

Apply the Limit

$$\frac{3(\infty)^2 - 5(\infty) + 1}{5(\infty)^2 - 3(\infty) + 1} \Rightarrow \frac{\infty + 1}{\infty + 1} \Rightarrow \frac{\infty}{\infty}$$

Hence it is $\frac{\infty}{\infty}$ form so $\times$ ing & $\div$ ing

Numerator and Denominator by "x^2"

$$\lim_{x\to\infty} \frac{\frac{3x^2 - 5x + 1}{x^2}}{\frac{5x^2 - 3x + 1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{5x}{x^2} + \frac{1}{x^2}}{\frac{5x^2}{x^2} - \frac{3x}{x^2} - \frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3 - \frac{5}{x} + \frac{1}{x^2}}{5 - \frac{3}{x} + \frac{1}{x^2}}$$

Apply the Limit

$$=> \frac{\lim_{x\to\infty}[\,3-\frac{5}{x}+\frac{1}{x^2}\,]}{\lim_{x\to\infty}[5-\frac{3}{x}+\frac{1}{x^2}\,]}$$

$$=\frac{3-\frac{5}{\infty}+\frac{1}{\infty^2}}{5-\frac{3}{\infty}+\frac{1}{\infty^2}}$$

$$=\frac{3-0+0}{5-0+0}$$

$$\boxed{\frac{3}{5}\ Ans.}$$

4. $\lim_{x\to\infty}\frac{3x^3-4x^2+5x-1}{9x^3+2x^2-6x+11}$

$Sol:-$

$$\lim_{x\to\infty}\frac{3x^3-4x^2+5x-1}{9x^3+2x^2-6x+11}$$

Apply the Limit

$$=\frac{3(\infty)^3-4(\infty)^2+5(\infty)-1}{9(\infty)^3+2(\infty)^2-6(\infty)+11} => \frac{\infty-1}{\infty+11} => \frac{\infty}{\infty}$$

Hence it is $\frac{\infty}{\infty}$ form so × ing & ÷ing

Numerator and Denominator by "x^3"

$$=\lim_{x\to\infty}\frac{\frac{3x^3-4x^2+5x-1}{x^3}}{\frac{9x^3+2x^2-6x+11}{x^3}}$$

$$=\lim_{x\to\infty}\frac{\frac{3\cancel{x^3}}{\cancel{x^3}}-\frac{4\cancel{x^2}}{x^{\cancel{3}}}+\frac{5\cancel{x}}{x^{\cancel{3}}}-\frac{1}{x^3}}{\frac{9\cancel{x^3}}{\cancel{x^3}}-\frac{2\cancel{x^2}}{x^{\cancel{3}}}-\frac{6\cancel{x}}{x^{\cancel{3}}}-\frac{11}{x^3}}$$

$$= \lim_{x\to\infty} \frac{3 - \frac{4}{x} + \frac{5}{x^2} - \frac{1}{x^3}}{9 - \frac{2}{x} - \frac{6}{x^2} + \frac{11}{x^3}}$$

Apply the Limit

$$=> \frac{\lim_{x\to\infty}[\, 3 - \frac{4}{x} + \frac{5}{x^2} - \frac{1}{x^3}\,]}{\lim_{x\to\infty}[\, 9 - \frac{2}{x} - \frac{6}{x^2} + \frac{11}{x^3}\,]}$$

$$= \frac{3 - \frac{4}{\infty} + \frac{5}{\infty^2} - \frac{1}{\infty^3}}{9 - \frac{2}{\infty} - \frac{6}{\infty^2} + \frac{11}{\infty^3}}$$

$$= \frac{3-0+0-0}{9-2-2+0} => \frac{\cancel{3}}{\cancel{9}}$$

$$\boxed{\frac{1}{3}\ Ans.}$$

- **Limit of Trigonometric Function :-**

1. $\lim_{x\to 0} \cos x = \cos 0 = 1$
2. $\lim_{x\to 0} \sin x = sin 0 = 0$
3. $\lim_{x\to 0} tan\ x = \lim_{x\to 0} \frac{sinx}{cosx} = \frac{0}{1} = 0$
4. $\lim_{x\to 0} \frac{sinx}{x} = 1$

- **Questions :-**

1. $\lim_{x\to 0} \frac{1+\sin x + sin2x}{\sqrt{4+\cos^2 x}}$

$Sol: -$

$$\lim_{x\to 0} \frac{1+\sin x + sin2x}{\sqrt{4+\cos^2 x}}$$

2. $\lim_{x\to 90°} \frac{x}{cosx}$

$Sol: -$

$$\lim_{x\to 90°} \frac{x}{cosx}$$

Apply the Limit

$$= \frac{\lim_{x\to 0}[\,1+\sin x+sin2x]}{\lim_{x\to 0}[\,\sqrt{4+\cos^2 x}]}$$

$$= \frac{[\,1+\sin(0)+sin2(0)]}{[\,\sqrt{4+\cos^2(0)}]}$$

$$= \frac{1+0+0}{\sqrt{4+1}}$$

$$\boxed{\frac{1}{\sqrt{5}}\ \ Ans.}$$

3. $\lim_{x\to 0}\frac{1}{xcotx}$

$Sol:-$

$$\lim_{x\to 0}\frac{1}{xcotx}$$

$$= \lim_{x\to 0}\frac{1}{x}\cdot\frac{1}{cotx} \qquad \because \frac{1}{cotx} = tanx$$

$$= \lim_{x\to 0}\frac{1}{x}\cdot\tan x \qquad \because tanx = \frac{\sin x}{\cos x}$$

$$= \lim_{x\to 0}\frac{1}{x}\cdot\frac{sinx}{cosx}$$

$$= \lim_{x\to 0}\frac{sinx}{x}\cdot\frac{1}{cosx} \qquad \because \lim_{x\to 0}\frac{sinx}{x} = 1$$

Apply the Limit

$$=> \lim_{x\to 0}\frac{sinx}{x}\cdot\lim_{x\to 0}\frac{1}{cosx}$$

Apply the Limit

$$= \frac{\lim_{x\to 90^\circ} x}{\lim_{x\to 90^\circ} cosx}$$

$$= \frac{90^\circ}{\cos 90^\circ} \qquad \because \cos 90^\circ = 0$$

$$= \frac{90^\circ}{0} \qquad => \text{undefined } (\infty)$$

Limit does not exist.

4. $\lim_{x\to 0}\frac{sinpx}{qx}$

$Sol:-$

$$\lim_{x\to 0}\frac{sinpx}{qx}$$

$$= \frac{1}{q}\lim_{x\to 0}\frac{sinpx}{x}$$

Multiplying & dividing p on both sides

$$= \frac{p}{q}\lim_{x\to 0}\frac{sinpx}{px}$$

Apply the Limit

$$=> \frac{p}{q}\lim_{x\to 0}\frac{sinpx}{px} \qquad \because \lim_{x\to 0}\frac{sinpx}{px} = 1$$

$$=> \frac{p}{q} = 1$$

$$\boxed{\frac{p}{q} = 1\ \ Ans.}$$

$$= 1.\frac{1}{\cos(0)}$$

$$= 1 \times \frac{1}{1}$$

$\boxed{1 \; Ans.}$

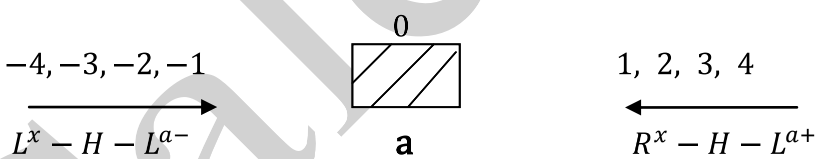
- **Left Hand Limit & Right Hand Limit : -**

If $x \longrightarrow a$ through values of "x" greater then "a", we say that "x" approaches "a" through the right. In symbols we write:

$$x \longrightarrow a + 0 \qquad \textbf{OR} \qquad x \longrightarrow a^{+} .$$

Similarly if $x \longrightarrow a$ through values of "x" less then "a" we say that "x" approaches "a" through the left. In symbols we write:

$$x \longrightarrow a - 0 \qquad \textbf{OR} \qquad x \longrightarrow a^{-} .$$



- **Questions :-**

  - Find the Limits :

1. $\lim\limits_{x \to 2-0} \frac{\sqrt{4-x^2}}{\sqrt{6-5(x)+x^2}}$

$Sol : -$

$$\lim_{x \to 2-0} \frac{\sqrt{4-x^2}}{\sqrt{6-5(x)+x^2}}$$

Using the result for $L - H - L$

$$\boxed{\lim_{x \to 2-0} f(x) = \lim_{h \to 0} (2-h)}$$

$$=> \lim_{h \to 2-0} f(2-h) = \lim_{h \to 0} \frac{\sqrt{4-(2-h)^2}}{\sqrt{6-5(2-h)+(2-h)^2}}$$

$$=> \lim_{h \to 2-0} f(2-h) = \lim_{h \to 0} \frac{\sqrt{4-(4-4h+h^2)}}{\sqrt{6-10+5h+4-4h+h^2}}$$

$$=> \lim_{h \to 2-0} f(2-h) = \lim_{h \to 0} \frac{\sqrt{\cancel{4}-\cancel{4}+4h-h^2)}}{\sqrt{\cancel{6}-1\cancel{0}+h+\cancel{4}+h^2}}$$

$$=> \lim_{h \to 2-0} f(2-h) = \lim_{h \to 0} \frac{\sqrt{4h-h^2}}{\sqrt{h+h^2}}$$

$$=> \lim_{h \to 2-0} f(2-h) = \lim_{h \to 0} \frac{\sqrt{\cancel{h}(4-h)}}{\sqrt{\cancel{h}(1+h)}}$$

$$=> \lim_{h \to 2-0} f(2-h) = \lim_{h \to 0} \frac{\sqrt{(4-h)}}{\sqrt{(1+h)}}$$

Apply the Limit

$$=> \lim_{h \to 2-0} f(2-h) = \frac{\lim_{h \to 0} \sqrt{(4-h)}}{\lim_{h \to 0} \sqrt{h(1+h)}}$$

$$=> \lim_{h \to 2-0} f(2-h) = \frac{\sqrt{(4-0)}}{\sqrt{(1+0)}}$$

$$=> \lim_{h \to 2-0} f(2-h) = \sqrt{\frac{4}{1}} \;=>\; \sqrt{4}$$

$$\boxed{\lim_{h \to 2-0} f(2-h) = 2 \;\; Ans.}$$

2. $\lim\limits_{x \to 1-0} \frac{\sqrt{1-x^2}}{1-x}$

$Sol:-$

$\lim\limits_{x \to 1-0} \frac{\sqrt{1-x^2}}{1-x}$

Using the result for $L - H - L$

$$\boxed{\lim_{x \to 1-0} f(x) = \lim_{h \to 0} (1-h)}$$

$$=> \lim_{h \to 1-0} f(1-h) = \lim_{h \to 0} \frac{\sqrt{1-(1-h)^2}}{1-(1-h)}$$

$$=> \lim_{h \to 1-0} f(1-h) = \lim_{h \to 0} \frac{\sqrt{1-(1^2+h^2-2h)}}{1-1+h}$$

$$=> \lim_{h \to 1-0} f(1-h) = \lim_{h \to 0} \frac{\sqrt{\cancel{1}-\cancel{1}-h^2+2h}}{\cancel{1}-\cancel{1}+h}$$

$$=> \lim_{h \to 1-0} f(1-h) = \lim_{h \to 0} \frac{\sqrt{2h-h^2}}{h}$$

$$=> \lim_{h \to 1-0} f(1-h) = \lim_{h \to 0} \frac{\sqrt{h^2(\frac{2}{h}-1)}}{h}$$

$$=> \lim_{h \to 1-0} f(1-h) = \lim_{h \to 0} \frac{\sqrt{h^{\cancel{2}}}\ \sqrt{(\frac{2}{h}-1)}}{h}$$

$$=> \lim_{h \to 1-0} f(1-h) = \lim_{h \to 0} \frac{\cancel{h}.\sqrt{(\frac{2}{h}-1)}}{\cancel{h}}$$

$$=> \lim_{h\to 1-0} f(1-h) = \lim_{h\to 0}\sqrt{\left(\frac{2}{h}-1\right)}$$

Now Apply the Limit

$$=> \lim_{h\to 1-0} f(1-h) = \sqrt{\left(\frac{2}{0}-1\right)} \qquad \because \ \frac{2}{0}\ \infty$$

$$=> \lim_{h\to 1-0} f(1-h) = \sqrt{(\infty-1)}$$

$$\boxed{\lim_{h\to 1-0} f(1-h) = \sqrt{(-1+\infty)}\ Ans.}$$

3. Find

   i. $f(2+0)$
   ii. $f(2-0)$

Where $f(x) = \frac{x^3-8}{x^2-4}$

i. $f(2+0)$

$Sol:-$

Given that

$$f(x) = \frac{x^3-8}{x^2-4}$$

Using the result for $R-H-L$

$$\boxed{\lim_{x\to 2+0} f(x) = \lim_{h\to 0}(2+h)}$$

$$=> \lim_{h\to 2+0} f(2+h) = \lim_{h\to 0}\frac{(2+h)^3-8}{(2+h)^2-4}$$

$$\boxed{(a+b)^3 = a^3+b^3+3a^2b+3ab^2}$$

$$\boxed{(a+b)^2 = a^2+b^2+2ab}$$

$$=> \lim_{h\to 2+0} f(2+h) = \lim_{h\to 0}\frac{(2)^3+(h)^3+3(2)^2(h)+3(2)(h)^2-8}{(2)^2+(h)^2+2(2)(h)-4}$$

$$=> \lim_{h\to 2+0} f(2+h) = \lim_{h\to 0}\frac{\cancel{8} + h^3 + 12h + 6h^2 - \cancel{8}}{\cancel{4} + h^2 + 4h - \cancel{4}}$$

$$=> \lim_{h\to 2+0} f(2+h) = \lim_{h\to 0}\frac{h^3 + 12h + 6h^2}{h^2 + 4h}$$

$$=> \lim_{h\to 2+0} f(2+h) = \lim_{h\to 0}\frac{\cancel{h}(h^2 + 12 + 6h)}{\cancel{h}(h + 4)}$$

$$=> \lim_{h\to 2+0} f(2+h) = \lim_{h\to 0}\frac{(h^2 + 12 + 6h)}{(h + 4)}$$

Apply the Limit

$$=> \lim_{h\to 2+0} f(2+h) = \frac{\lim_{h\to 0}[h^2 + 12 + 6h]}{\lim_{h\to 0}[h + 4]}$$

$$= \frac{(0)^2 + 12 + 6(0)}{(0) + 4} => \frac{\cancel{12}}{\cancel{4}}$$

$$\boxed{\lim_{h\to 2-0} f(2+h) = 3\ Ans.}$$

ii. $f(2-0)$

$Sol: -$

Given that

$$f(x) = \frac{x^3 - 8}{x^2 - 4}$$

Using the result for $L - H - L$

$$\boxed{\lim_{x\to 2-0} f(x) = \lim_{h\to 0}(2-h)}$$

$$=> \lim_{h\to 2-0} f(2-h) = \lim_{h\to 0}\frac{(2-h)^3 - 8}{(2-h)^2 - 4}$$

$$\boxed{(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2}$$

$$\boxed{(a-b)^2 = a^2 - b^2 - 2ab}$$

$$=> \lim_{h\to 2-0} f(2-h) = \lim_{h\to 0} \frac{(2)^3 - (h)^3 - 3(2)^2(h) + 3(2)(h)^2 - 8}{(2)^2 + (h)^2 - 2(2)(h) - 4}$$

$$=> \lim_{h\to 2-0} f(2-h) = \lim_{h\to 0} \frac{\cancel{8} - h^3 - 12h + 6h^2 - \cancel{8}}{\cancel{4} + h^2 - 4h - \cancel{4}}$$

$$=> \lim_{h\to 2-0} f(2-h) = \lim_{h\to 0} \frac{-h^3 - 12h + 6h^2}{h^2 - 4h}$$

$$=> \lim_{h\to 2-0} f(2-h) = \lim_{h\to 0} \frac{\cancel{h}(-h^2 - 12 + 6h)}{\cancel{h}(h-4)}$$

$$=> \lim_{h\to 2-0} f(2-h) = \lim_{h\to 0} \frac{-h^2 - 12 + 6h}{h-4}$$

Apply the Limit

$$=> \lim_{h\to 2-0} f(2-h) = \frac{\lim_{h\to 0}[-h^2 - 12 + 6h]}{\lim_{h\to 0}[h-4]}$$

$$= \frac{-(0)^2 - 12 + 6(0)}{(0) - 4} => \frac{\cancel{-12}}{\cancel{-4}}$$

$$\boxed{\lim_{h\to 2-0} f(2-h) = 3\ Ans.}$$

4. Find

i. $f(1+0)$
ii. $f(1-0)$

Where $f(x) = \frac{x^3-1}{x^2-1}$

i. $f(1+0)$

$Sol:-$

Given that

$$f(x) = \frac{x^3-1}{x^2-1}$$

Using the result for $R-H-L$

$$\boxed{\lim_{x\to 1+0} f(x) = \lim_{h\to 0}(1+h)}$$

$$=> \lim_{h\to 1+0} f(1+h) = \lim_{h\to 0}\frac{(1+h)^3-1}{(1+h)^2-1}$$

$$\boxed{(a+b)^3 = a^3+b^3+3a^2b+3ab^2}$$

$$\boxed{(a+b)^2 = a^2+b^2+2ab}$$

$$=> \lim_{h\to 1+0} f(1+h) = \lim_{h\to 0}\frac{(1)^3+(h)^3+3(1)^2(h)+3(1)(h)^2-1}{(1)^2+(h)^2+2(1)(h)-1}$$

$$=> \lim_{h\to 1+0} f(1+h) = \lim_{h\to 0}\frac{\cancel{1}+h^3+3h+3h^2-\cancel{1}}{\cancel{1}+h^2+2h-\cancel{1}}$$

$$=> \lim_{h\to 1+0} f(1+h) = \lim_{h\to 0}\frac{h^3+3h+3h^2}{h^2+2h}$$

$$=> \lim_{h\to 1+0} f(1+h) = \lim_{h\to 0}\frac{\cancel{h}(h^2+3+3h)}{\cancel{h}(h+2)}$$

$$=> \lim_{h\to 1+0} f(1+h) = \lim_{h\to 0}\frac{(h^2+3+3h)}{(h+2)}$$

Apply the Limit

$$=> \lim_{h\to 1+0} f(1+h) = \frac{\lim_{h\to 0}[h^2+3+36h]}{\lim_{h\to 0}[h+2]}$$

$$= \frac{(0)^2+3+3(0)}{(0)+2} \quad => \frac{3}{2}$$

$$\boxed{\lim_{h\to 1-0} f(1+h) = \frac{3}{2} \text{ Ans.}}$$

ii. $f(1-0)$

$Sol: -$

Given that

$$f(x) = \frac{x^3-1}{x^2-1}$$

Using the result for $R-H-L$

$$\boxed{\lim_{x\to 1-0} f(x) = \lim_{h\to 0}(1-h)}$$

$$=> \lim_{h\to 1-0} f(1-h) = \lim_{h\to 0}\frac{(1-h)^3-1}{(1-h)^2-1}$$

$$\boxed{(a-b)^3 = a^3-b^3-3a^2b+3ab^2}$$

$$\boxed{(a-b)^2 = a^2+b^2-2ab}$$

$$=> \lim_{h\to 1-0} f(1-h) = \lim_{h\to 0}\frac{(1)^3-(h)^3-3(1)^2(h)+3(1)(h)^2-1}{(1)^2+(h)^2-2(1)(h)-1}$$

$$=> \lim_{h\to 1-0} f(1-h) = \lim_{h\to 0}\frac{\cancel{1}-h^3-3h+3h^2-\cancel{1}}{\cancel{1}+h^2-2h-\cancel{1}}$$

$$=> \lim_{h\to 1-0} f(1-h) = \lim_{h\to 0}\frac{-h^3-3h+3h^2}{h^2-2h}$$

$$=> \lim_{h\to 1-0} f(1-h) = \lim_{h\to 0} \frac{\cancel{h}(-h^2-3h+3h)}{\cancel{h}(h-2)}$$

$$=> \lim_{h\to 1-0} f(1-h) = \lim_{h\to 0} \frac{(-h^2-3+3h)}{(h-2)}$$

Apply the Limit

$$=> \lim_{h\to 1-0} f(1-h) = \frac{\lim_{h\to 0}[-h^2-3+36h]}{\lim_{h\to 0}[h-2]}$$

$$= \frac{-(0)^2-3+3(0)}{(0)-2} \quad => \frac{\cancel{-}3}{\cancel{-}2}$$

$$\boxed{\lim_{h\to 1-0} f(1-h) = \frac{3}{2} \text{ Ans.}}$$

5. Find

i. $f(4+0)$
ii. $f(4-0)$

Where $f(x) = \frac{x^2-16}{x-4}$

i. $f(4+0)$

$Sol:-$

Given that

$$f(x) = \frac{x^2-16}{x-4}$$

Using the result for $R-H-L$

$$\boxed{\lim_{x\to 4+0} f(x) = \lim_{h\to 0}(4+h)}$$

$$=> \lim_{h\to 4+0} f(4+h) = \lim_{h\to 0} \frac{(4+h)^2-16}{(4+h)-4}$$

$$\boxed{(a+b)^2 = a^2+b^2+2ab}$$

$$=> \lim_{h\to 4+0} f(4+h) = \lim_{h\to 0} \frac{(4)^2 + (h)^2 + 2(4)(h) - 16}{4+h-4}$$

$$=> \lim_{h\to 4+0} f(4+h) = \lim_{h\to 0} \frac{16 + h^2 + 8h - 16}{4 + h - 4}$$

$$=> \lim_{h\to 4+0} f(4+h) = \lim_{h\to 0} \frac{h^2 + 8h}{h}$$

$$=> \lim_{h\to 4+0} f(4+h) = \lim_{h\to 0} \frac{h(h+8)}{h}$$

$$=> \lim_{h\to 4+0} f(4+h) = \lim_{h\to 0} (h+8)$$

Apply the Limit

$$=> \lim_{h\to 4+0} f(4+h) = \lim_{h\to 0} (h+8)$$

$$=> \lim_{h\to 4+0} f(4+h) \quad => 0+8$$

$$=> \lim_{x\to 4+0} f(4+h) = 8 \text{ Ans.}$$

ii. $f(4-0)$

$Sol: -$

Given that

$$f(x) = \frac{x^2 - 16}{x - 4}$$

Using the result for $L - H - L$

$$\lim_{x\to 4-0} f(x) = \lim_{h\to 0} (4-h)$$

$$=> \lim_{h\to 4-0} f(4-h) = \lim_{h\to 0} \frac{(4-h)^2 - 16}{(4-h) - 4}$$

$$(a-b)^2 = a^2 + b^2 - 2ab$$

$$=> \lim_{h\to 4-0} f(4 \pm h) = \lim_{h\to 0} \frac{(4)^2 + (h)^2 - 2(4)(h) - 16}{4 - h - 4}$$

$$=> \lim_{h\to 4-0} f(4 - h) = \lim_{h\to 0} \frac{\cancel{16} + h^2 - 8h - \cancel{16}}{\cancel{4} - h - \cancel{4}}$$

$$=> \lim_{h\to 4-0} f(4 - h) = \lim_{h\to 0} \frac{h^2 - 8h}{-h}$$

$$=> \lim_{h\to 4-0} f(4 - h) = \lim_{h\to 0} \frac{\cancel{h}(h - 8)}{-\cancel{h}}$$

$$=> \lim_{h\to 4-0} f(4 - h) = \lim_{h\to 0} -(h - 8)$$

$$=> \lim_{h\to 4-0} f(4 - h) = \lim_{h\to 0} (-h + 8)$$

Apply the Limit

$$=> \lim_{h\to 4-0} f(4 - h) = \lim_{h\to 0} (-h + 8)$$

$$=> \lim_{h\to 4-0} f(4 - h) \quad => (-0 + 8)$$

$$\boxed{=> \lim_{h\to 4-0} f(4 - h) = 8 \text{ Ans.}}$$

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The End of Week # 04