Week # 03

Functions and Graphs

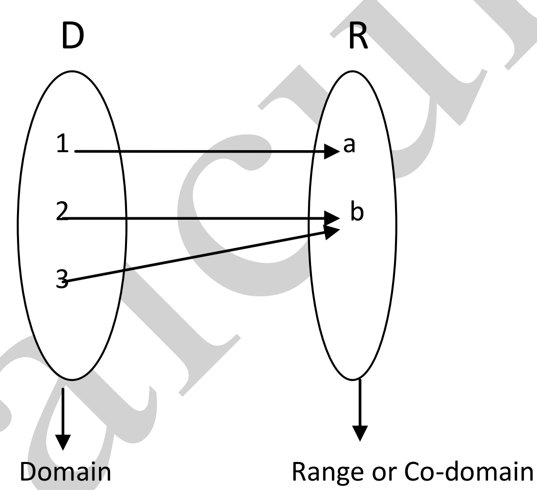
- **FUNCTION:-**

A function from a set "D" to a set "R" is a rule that assigns a unique element $f(x)$ in "R" to each element "x" in "D".

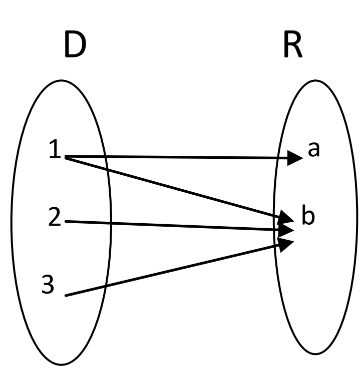
OR

Every input which exactly only one output is called function.

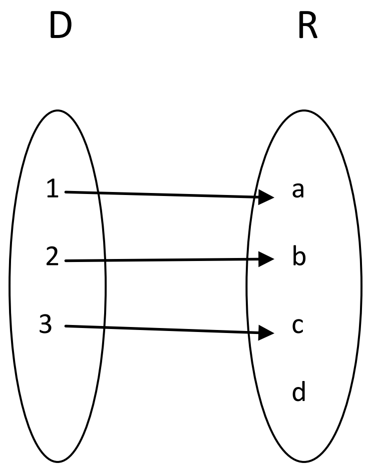
$D = \{ 1,2,3 \}$
$R = \{ a, b \}$



- In the above function Domain is equal to set "D".
- Domain is non-repeated.
- So it is a function.



❖ It is not a function because Domain is repeated.



❖ It is a function.

- **Types of functions:-**

1. **Linear equations:-**

$ax + b = 0\ ,\quad a \neq 0$ Is called Linear Equations.

- **Linear Function:-**

$F(x) = ax + b = 0\ ,\ a \neq 0$ is called Linear Function.

❖ $f(x) = 1x + 0 \qquad$ =>> $f(x) = x^1$

❖ $f(x) = 1x + 10 \qquad$ =>> $f(x) = x + 1$

❖ $f(x) = \sqrt{2}x^1 + \sqrt{3} \qquad$ =>> $f(x) = \sqrt{2}\text{x} + \sqrt{3}$

- **Questions :-**

1. $f(x) = x$

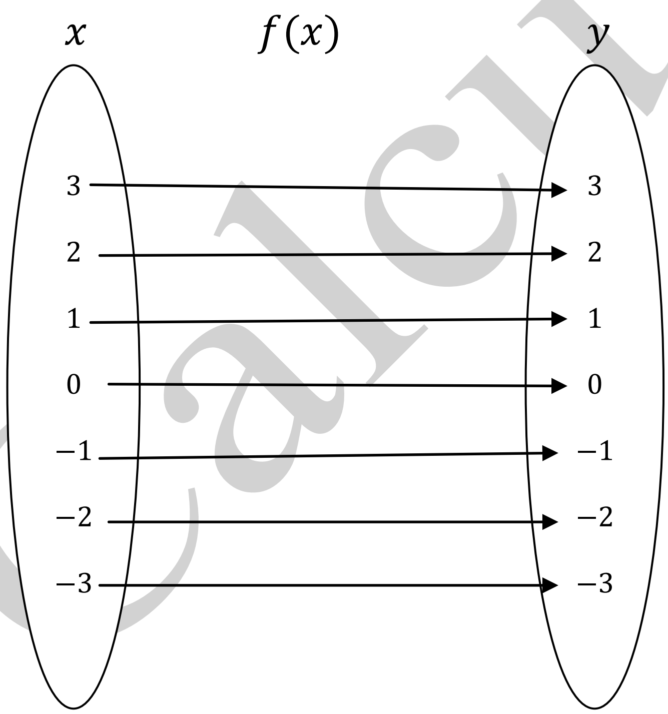
$Sol: -$

- $y = x$

Domain + Input + Pre-image + Independent Variable

x	-3	-2	-1	0	1	2	3	4	5
$y = x$	-3	-2	-1	0	1	2	3	4	5

Range + Output + Image + Dependent Variable



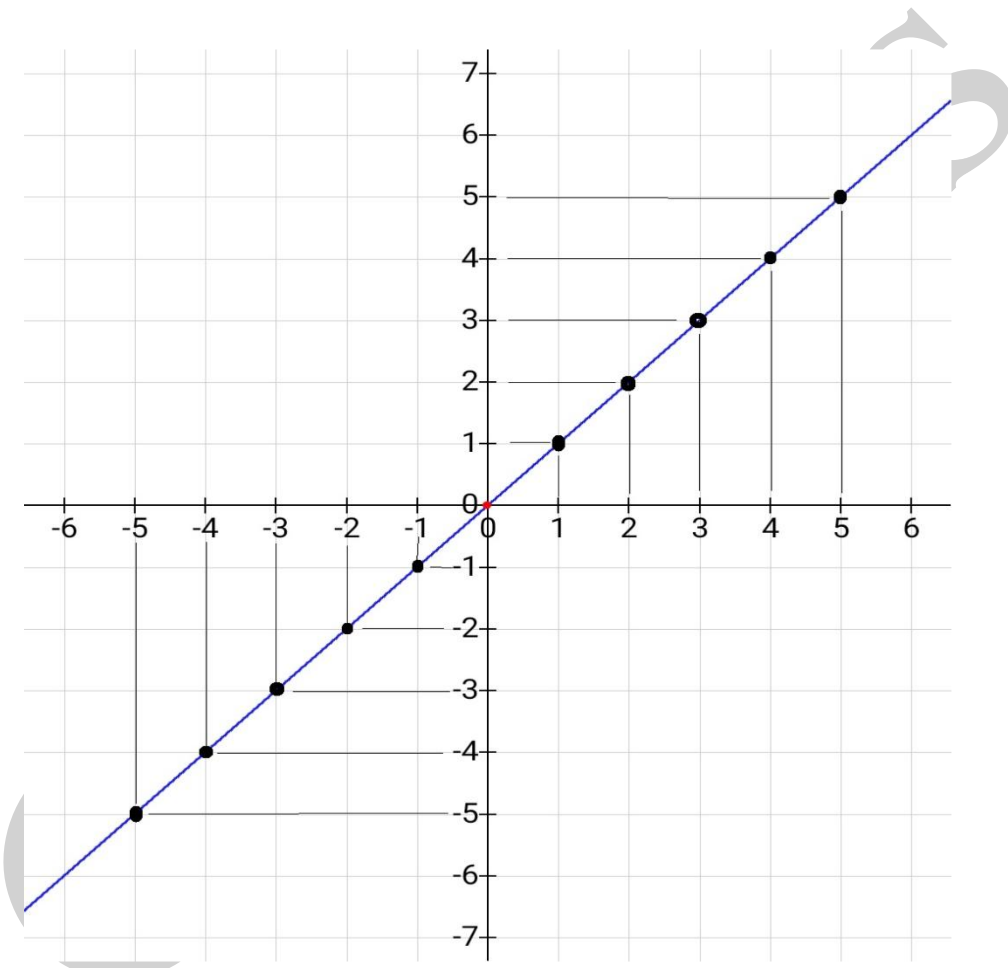
- Domain
  $D = x \in IR$ (-∞ , ∞)

- Range
  R= $y \in IR$ (-∞ , ∞)

- Graph of Linear function.

i. $f(x) = x.$
$x \in \{-3, -2, -1, 0, 1, 2, 3\}$

x	-3	-2	-1	0	1	2	3	4	5
$y = x$	-3	-2	-1	0	1	2	3	4	5

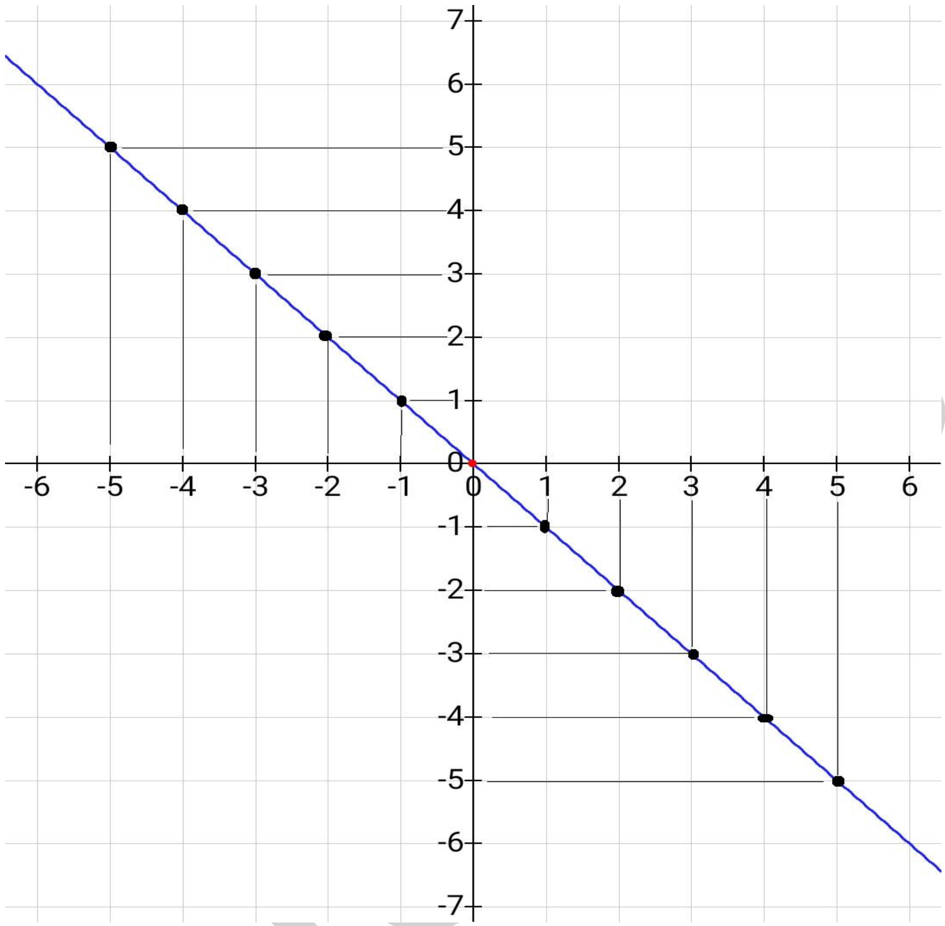


❖ The Graph of Linear Equation will give straight line.

ii. $F(x) = -x$

$x \in \{-3, -2, -1, 0, 1, 2, 3\}$

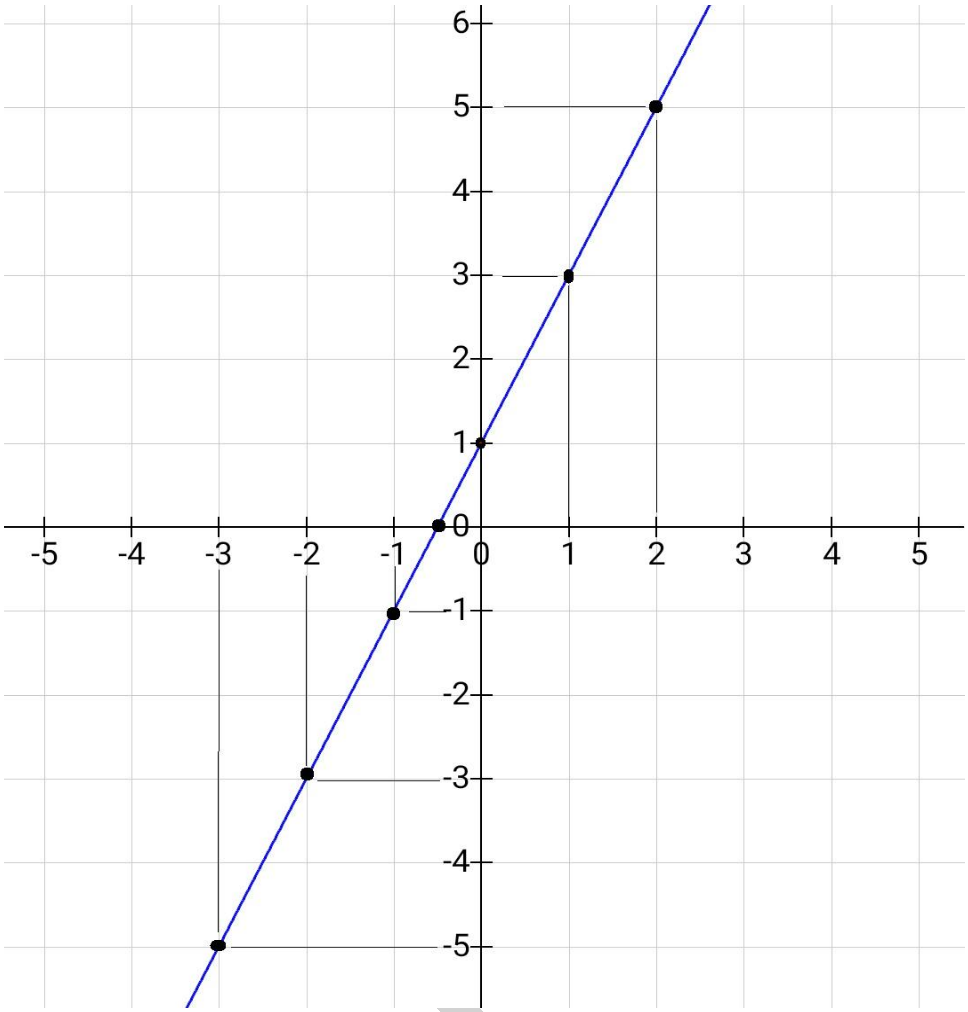
x	-3	-2	-1	0	1	2	3	4	5
$y = -x$	3	2	1	0	-1	-2	-3	-4	-5



iii. $f(x) = 2x + 1$

$x \in \{ -3, -2, -1, 0, 1, 2, 3 \}$

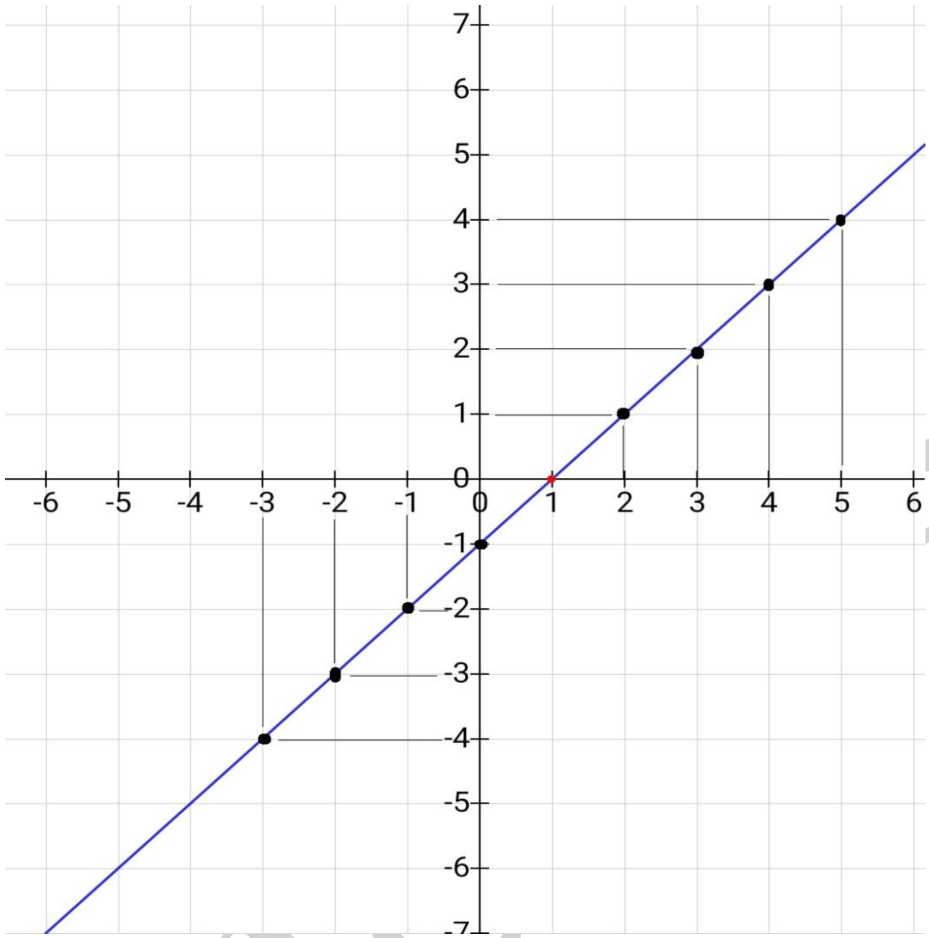
x	-3	-2	-1	0	1	2	3	4
$y = 2x + 1$	-5	-3	-1	1	3	5	7	9



iv. $f(x) = x - 1$

$x \in \{ -3, -2, -1, 0, 1, 2, 3 \}$

x	-3	-2	-1	0	1	2	3	4	5
$y = x - 1$	-4	-3	-2	-1	0	1	2	3	4



2. **Quadratic Equations:-**

$ax^2 + bx + c = 0$, $a \neq 0$ is called quadratic equations.

- **Quadratic Function:-**

$f(x) = y = ax^2 + bx + c = 0$, $a \neq 0$ is called quadratic function.

- Domain
$D = x \in IR$

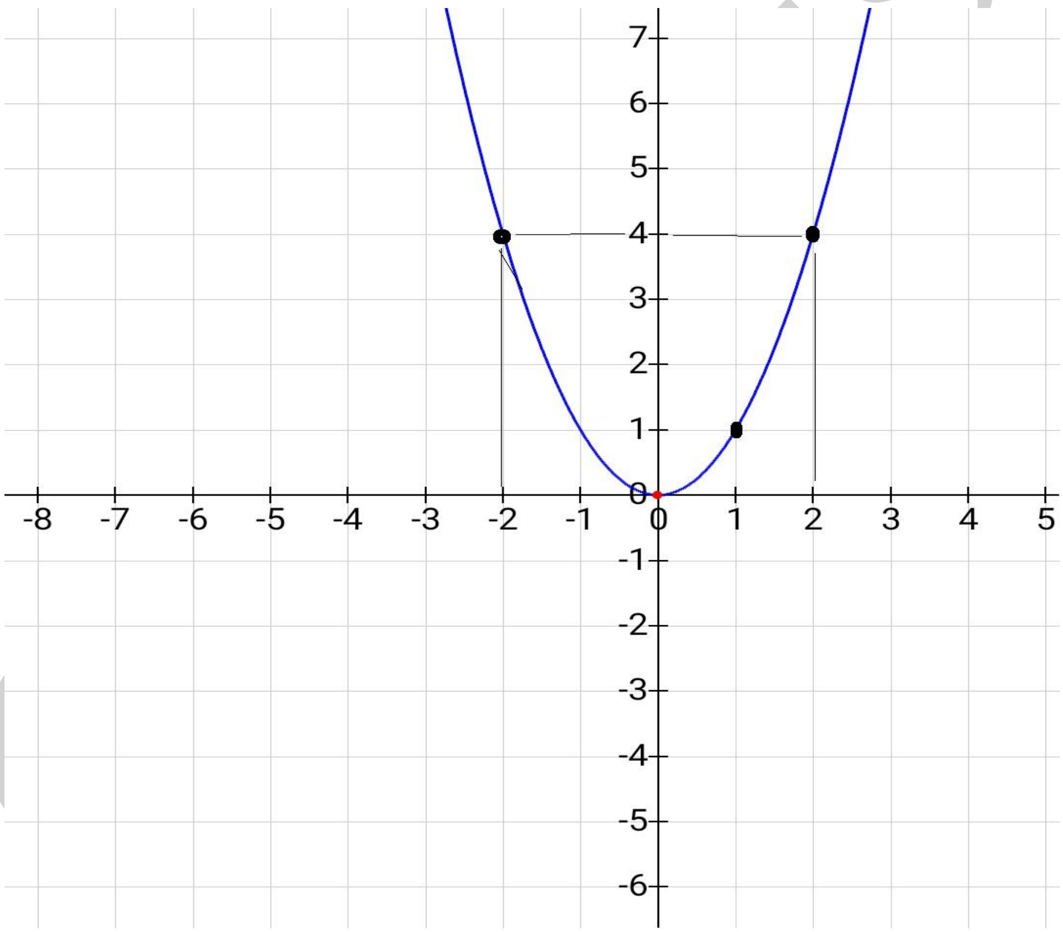
- Range
R= $y \in IR$

- **Questions:-**

i. $f(x) = x^2$

$x \in \{ -3, -2, -1, 0, 1, 2, 3 \}$

x	-3	-2	-1	0	1	2	3	4	5
$y = x^2$	9	4	1	0	1	4	9	16	25

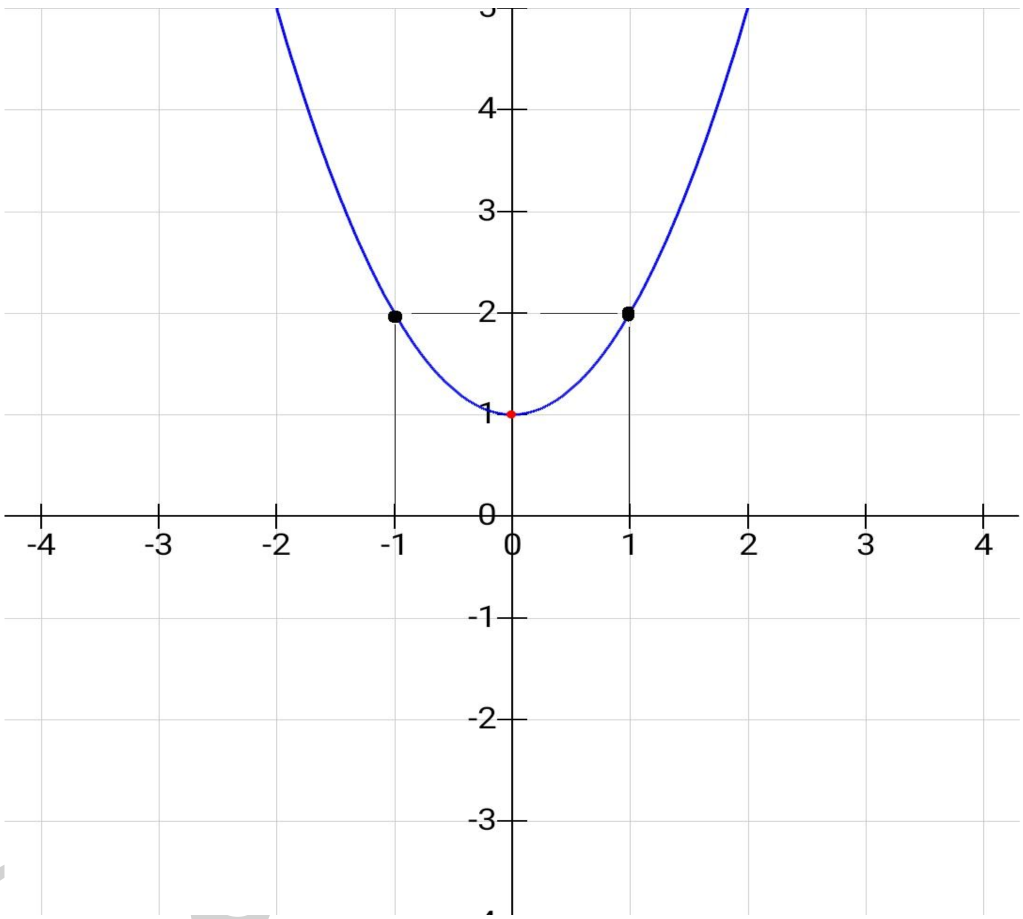


- The Quadratic graph will be Always Parabola

ii. $x^2 + 1$

$x \in \{ -3, -2, -1, 0, 1, 2, 3 \}$

x	-3	-2	-1	0	1	2	3	4	5
$y = x^2$+1	10	5	2	1	2	5	10	17	26

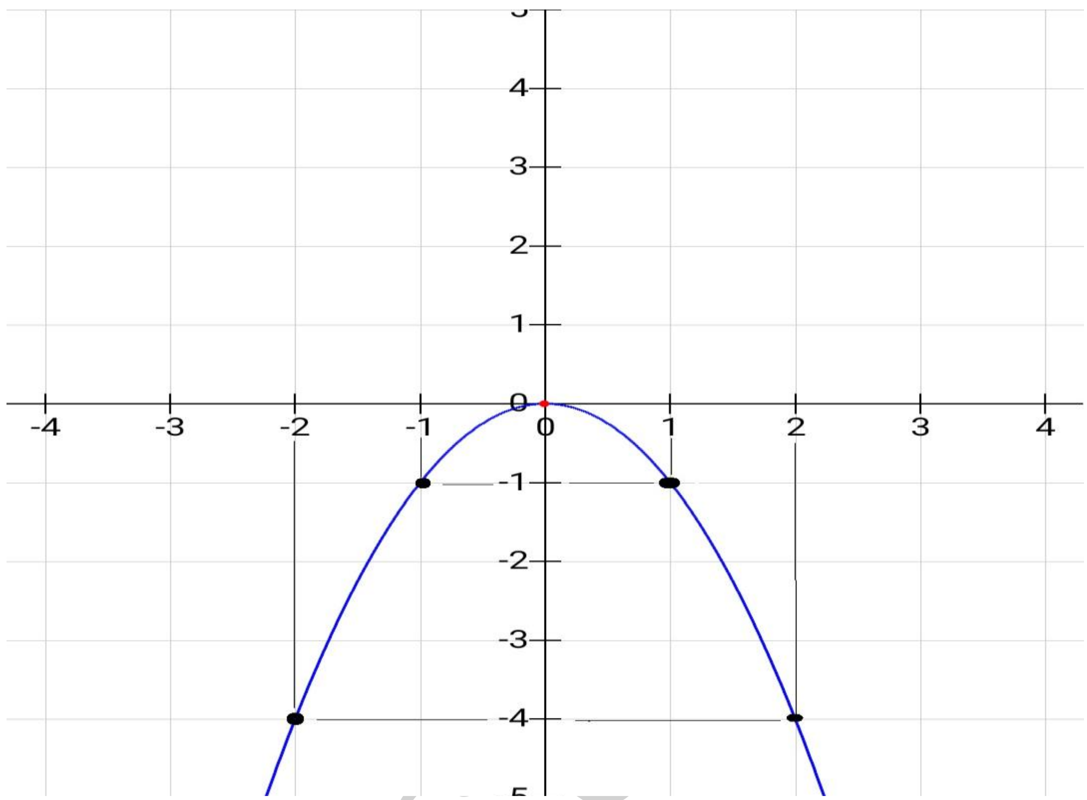


- The Quadratic graph will be Always Parabola

iii. $-x^2$

$x \in \{ -3, -2, -1, 0, 1, 2, 3 \}$

x	-3	-2	-1	0	1	2	3	4	5
$y = -x^2$	-9	-4	-1	0	-1	-4	-9	-16	-25



- The Quadratic graph will be Always Parabola

iv. $-x^2 - 1$

$x \in \{ -3, -2, -1, 0, 1, 2, 3 \}$

x	-3	-2	-1	0	1	2	3
$y = -x^2 - 1$	-10	-5	-2	-1	-2	-5	-10

- **Cube function:-**

$f(x) = x^3$ **is called cube function.**

Domain
$D = x \in IR$

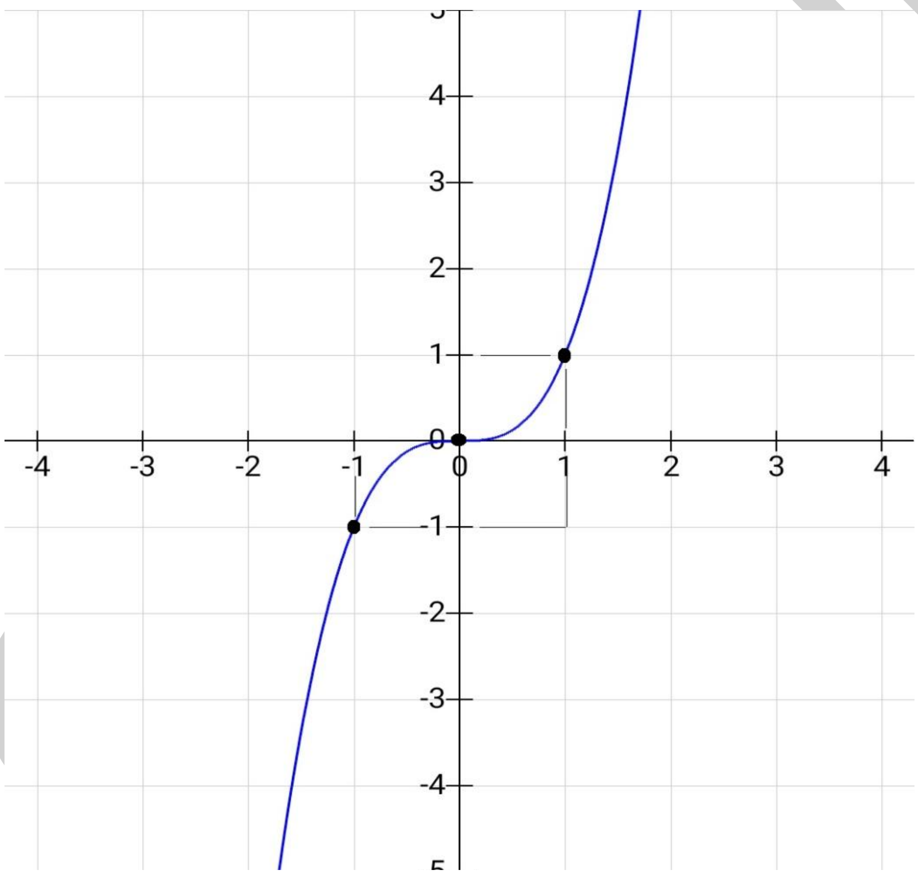
Range
R= $y \in IR$

- **Questions:-**

i. $f(x) = x^3$

$x \in \{ -3, -2, -1, 0, 1, 2, 3 \}$

x	-3	-2	-1	0	1	2	3
$y = x^3$	-9	-8	-1	0	1	8	27



- **Even function :-**

$\boldsymbol{A\ function\ f(x)\ is\ said\ to\ be\ even\ function\ , if\ f(-x) = f(x).}$

- **Questions:-**

i. $F(x) = x^2$

$Sol:-$

$f(x) = x^2$

$put\ x = -x$

$f(-x) = (-x)^2$
$f(-x) = x^2$
$\boxed{f(-x) = f(x)\ Ans.}$ Even Function

ii. $F(x) = x^2 + 1$

$Sol:-$

$f(x) = x^2 + 1$

$put\ x = -x$

$f(-x) = (-x)^2 + 1$
$f(-x) = x^2 + 1$
$\boxed{f(-x) = f(x)\ Ans.}$ Even Function

- **Odd Function:-**

A function $f(x)$ is said to be even function, if $f(-x) = -f(x)$.

- **Questions:-**

i. $F(x) = x^3$

$Sol:-$

$f(x) = x^3$
$put\ x = -x$

$f(-x) = (-x)^3$

$f(-x) = -x^3$

$\boxed{f(-x) = -f(x)\ Ans.}$ Odd Function

- **Neither Function:-**

A function which is neither even nor odd.

- **Questions:-**

i. $f(x) = x + 1$

$Sol: -$

$f(x) = x + 1$

$put\ x = -x$

$f(-x) = (-x) + 1$

$f(-x) = -x + 1$

$\boxed{f(-x) = -x + 1\ Ans.}$ Neither Function

ii. $f(x) = x^2 + x$

$Sol: -$

$f(x) = x^2 + x$

$put\ x = -x$

$f(-x) = (-x^2) + x$

$f(-x) = x^2 - x$

$f(-x) = -(-x^2 + x)$

$\boxed{f(-x) = -(-x^2 + x)\ Ans.}$ Neither Function

- Show that whether the function is even, odd or neither.

i. $f(x) = 3$

$Sol: -$

$f(x) = 3$

$put\ x = -x$

ii. $f(x) = x^{-5}$

$Sol: -$

$f(x) = x^{-5}$

$put\ x = -x$

$f(-x) = 3$

$f(-x) = f(x)$

$\boxed{f(-x) = f(x)\ Ans.}$ Even Function

iii. $f(x) = x^2 + 1$

$Sol:-$

$f(x) = x^2 + 1$

$put\ x = -x$

$f(-x) = (-x^2) + 1$

$f(-x) = x^2 + 1$

$\boxed{f(-x) = f(x)\ Ans.}$ Even Function

$f(-x) = (-x)^{-5}$

$f(-x) = -x^{-5}$

$\boxed{f(-x) = -f(x)\ Ans.}$ Odd Function

iv. $f(x) = x^2 + x$

$Sol:-$

$f(x) = x^2 + x$

$put\ x = -x$

$f(-x) = (-x^2) - x$

$f(-x) = x^2 - x$

$f(-x) = -(-x^2 + x)$

$f(-x) \neq f(x)$

$f(-x) \neq -f(x)$

Neither Function

v . . $g(x) = x^3 + x$

$Sol:-$

$g(x) = x^3 + x$

$put\ x = -x$

$g(-x) = (-x)^3 - x$

$g(-x) = -x^3 - x$

$g(-x) = -(x^3 + x)$

$\boxed{g(-x) = -\ g(x)\ Ans.}$ Even Function

vi. $g(x) = x^4 + 3x^2 - 7$

$Sol:-$

$g(x) = x^4 + 3x^2 - 7$

$put\ x = -x$

$g(-x) = (-x^4) + 3(-x^2) - 7$

$g(-x) = x^4 + 3x^2 - 7$

$\boxed{g(-x) = g(x)\ Ans.}$ Even Function

vii. $g(x) = \frac{1}{x^2-1}$

$Sol:-$

$g(x) = \frac{1}{x^2-1}$

$put\ x = -x$

$g(-x) = \frac{1}{(-x)^2-1}$

$g(-x) = \frac{1}{x^2-1}$

$\boxed{g(-x) = g(x)\ Ans.}$ Even function

viii. $h(t) = \frac{1}{t-1}$

$Sol:-$

$h(t) = \frac{1}{t-1}$

$put\ t = -t$

$h(-t) = \frac{1}{-t-1}$

$h(-t) = \frac{1}{-(t+1)}$

$h(-t) \neq h(t)$

$h(-t) \neq -h(t)$

neither function

ix. $h(t) = |t^3|$

$Sol:-$

$h(t) = |t^3|$

$put\ t = -t$

$h(-t) = |-t^3|$

$h(-t) = |-1 \times t^3|$

$h(-t) = |-1|\,.|t^3| \qquad \because |-1| = 1$

$h(-t) = |t^3|$

$\boxed{h(-t) = h(t)\ Ans.}$ Even Function

x. $h(t) = 2t + 1$

$Sol:-$

$h(t) = 2t + 1$

$put\ t = -t$

$h(-t) = 2(-t) + 1$

$h(-t) = -2t + 1$

$h(-t) = -(2t - 1)$

$h(-t) \neq h(t)$

$h(-t) \neq -h(t)$

Neither Function

ix. $h(t) = 2|t| + 1$

$Sol:-$

$h(t) = 2|t| + 1$

$put\ t = -t$

$h(-t) = 2|-t| + 1$

$h(-t) = 2|-1 \times t| + 1$

$h(-t) = 2|-1|.|t| + 1 \qquad \because\ |-1| = 1$

$h(-t) = 2|t| + 1$

$\boxed{h(-t) = h(t)\ Ans.}$ Even Function

- **Composition or Composite Function:-**

Composition is another method for combining functions.

❖ **Definition:-**

If f(x) and g(x) are two functions are composite function

(fog) Is defined by

$(fog)(x) = f(g(x)).$

$In\ fog$ "o" $represents\ circle\ or\ composite$

- **Questions:-**

i. $F(x) = \sqrt{x},\ \ g(x) = x + 1$

Find………

(a). $(fog)(x)$

(b). $(gof)(x)$

(c). $(fof)(x)$

(d). $(gog)(x)$

(a). $(fog)(x)$

$Sol:-$

$f(x) = \sqrt{x}$

$f(g(x)) = \sqrt{(g(x))}$

$(fog)(x) = \sqrt{x+1}$

$\boxed{(fog)(x) = \sqrt{x+1}\ \ Ans.}$

(b). $(gof)(x)$

$Sol:-$

$g(x) = x + 1$

$g(f(x)) = f(x) + 1$

$\boxed{(gof)(x) = \sqrt{x} + 1\ \ Ans.}$

(c). $(fof)(x)$

$Sol:-$

$f(x) = \sqrt{x}$

$f\big(f(x)\big) = \sqrt{\sqrt{x}}$

$(fof)(x) = \left(x^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$(fof)(x) = x^{\frac{1}{4}}$

$\boxed{(fof)(x) = x^{\frac{1}{4}}\ \ Ans.}$

(d). $(gog)(x)$

$Sol:-$

$g(x) = x + 1$

$g(g(x)) = g(x) + 1$

$(gog)(x) = x + 1 + 1$

$(gog)(x) = x + 2$

$\boxed{(gog)(x) = x + 2\ \ Ans.}$

ii. $f(x) = x^2 - 2$

$g(x) = x + 3$

$Sol:-$

$Find\ fog\ gof$

$fog = ?$

$f(x) = x^2 - 2$

$f(g(x)) = (g(x)^2) - 2$

$(fog)(x) = (x+3)^2 - 2$

$(fog)(x) = x^2 + 6x + 9 - 2$

$(fog)(x) = x^2 + 6x + 7$

$\boxed{(fog)(x) = (x^2 + 6x + 7\ Ans.}$

$gof = ?$

$g(x) = x + 3$

$g(f(x)) = f(x) + 3$

$(gof)(x) = x^2 - 2 + 3$

$(gof)(x) = x^2 + 1$

$\boxed{(gof)(x) = x^2 + 1\ Ans.}$

iii. $f(x)x + 5\ , g(x) = \mathrm{x}^2 - 3$

Find………..b

(a). $(fog)(\mathrm{x})$

(b). $(gof)(\mathrm{x})$

(c). $(fof)(\mathrm{x})$

(d). $(gog)(x)$

(a). $(fog)(\mathrm{x})$ $Sol:-$	(b). $(gof)(x)$ $Sol:-$

$f(x) = x + 5$

$f(g(x)) = g(x) + 5$

$(fog)(x) = x^2 - 3 + 5$

$(fog)(x) = x^2 + 2$

$\boxed{(fog)(x) = x^2 + 2\ Ans.}$

$g(x) = x^2 - 3$

$g(f(x)) = (f(x)^2) - 3$

$(gof)(x) = (x + 5)^2 - 3$

$(gof)(x) = (x^2 + 2(x)(5) + 25 - 3$

$(gof)(x) = x^2 + 10x + 22$

$\boxed{(gof)(x) = x^2 + 10x + 22\ Ans.}$

(c). $fof(x)$

$Sol: -$

$f(x) = x + 5$

$f(f(x)) = f(x) + 5$

$fof(x) = x + 5 + 5$

$fof(x) = x + 10$

$\boxed{fof(x) = x + 10\ Ans.}$

(d). $gog(x)$

$Sol: -$

$g(x) = x^2 - 3$

$g(g(x)) = (g(x)^2 - 3$

$gog(x) = (x^2 - 3)^2 - 3$

$gog(x) = (x^2)^2 - 2(x^2)(3) + 9 - 3$

$gog(x) = x^4 - 6x^2 + 6$

$\boxed{gog(x) = x^4 - 6x^2 + 6\ \ Ans.}$

iv. $f(x) = x - 1\,, g(x) = \frac{1}{x+1}$

(a) $fog(x)$

(b) $Gof(x)$

(c) $fof(x$

(d) $gog(x)$

(a) $fog(x)$

$Sol:-$

$f(x) = x - 1$

$g(x) = \frac{1}{x+1}$

$F(g(x)) = (g(x)) - 1$

$(fog)(x) = \frac{1}{x+1} - 1$

$(fog)(x) = \frac{1-1}{x+1}$

$(fog)(x) = \frac{0}{x+1}$

$\boxed{(fog)(x) = \frac{0}{x+1}\ Ans.}$

(b) $gof(x)$

$Sol:-$

$g(x) = \frac{1}{x+1}$

$g(f(x)) = \frac{1}{(f(x))+1}$

$(gof)(x) = \frac{1}{(x-1)+1}$

$(gof)(x) = \frac{1}{x - \cancel{1} + \cancel{1}}$

$\boxed{(gof)(x) = \frac{1}{x}\ Ans.}$

(c) $fof(x)$

$Sol:-$

$f(x) = x - 1$

$f(f(x)) = f(x) - 1$

$(fof)(x) = (x - 1) - 1$

(d) $gog(x)$

$Sol:-$

$g(x) = \frac{1}{x+1}$

$g(g(x)) = \frac{1}{g(x)+1}$

$g(g(x)) = \frac{1}{\frac{1}{x+1}+1}$

$(fof)(x) = x - 1 - 1$

$(fof)(x) = x - 2$

$\boxed{(fof)(x) = x - 2 \ Ans.}$

$g(g(x)) = \dfrac{1}{\frac{1+1}{x+1}}$

$gog(x) = \dfrac{1}{\frac{2}{x+1}}$

$\boxed{gog(x) = \frac{2}{x+1} Ans.}$

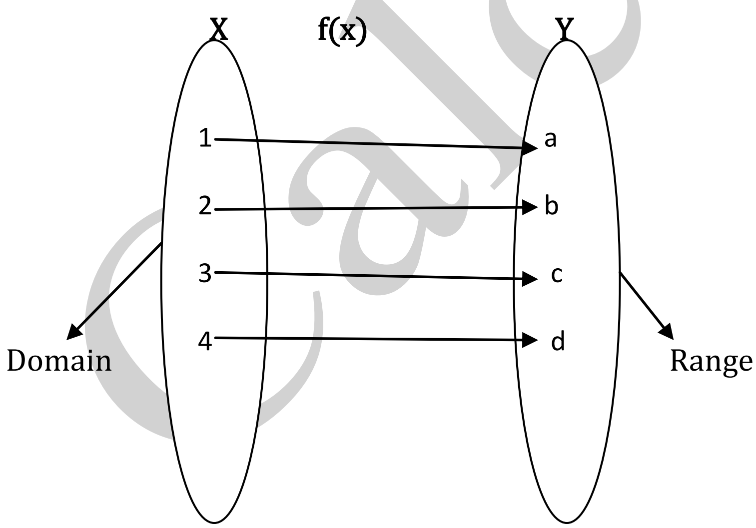
- **Inverse Function :-**

For any inverse function. The function must be bijective function.

- **Bijective Function :-**

i. One-to-one function

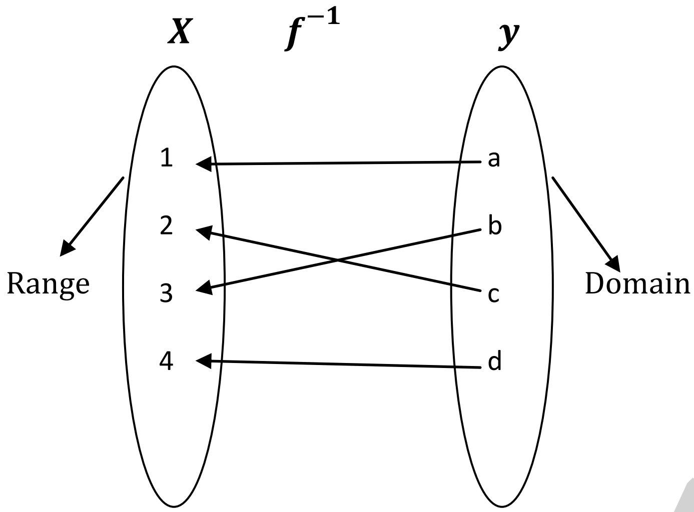
ii. On-to-function



i. Domain of $f(x)$ = Range of $f^{-1}(x)$

ii. Range of $f(x)$ = Domain of $f^{-1}(x)$

$R = \{ (1, a), (2, c), (3, b), (4, d)\}$



$R = \{ (a\ 1), (c, 2\ ), (b, 3), (d, 3)\}$

- **Questions:-**

i. $f(x) = x + 5$ find $f^{-1} =?$

Sol:

Step 1:-

Suppose $f(x) = y$

$x = \ \boldsymbol{f^{-1}}(\boldsymbol{y})$

$y = x + 5$

Step 2:-

$y = x + 5$

$y - 5 = x + \cancel{-5} + \cancel{5}$

$y - 5 = x$

Step 3:-

Replace "x" by $\boldsymbol{f^{-1}}(\boldsymbol{y})$

$\boldsymbol{f^{-1}(y) = y - 5}$

Step 4:-

$$f^{-1}(y) = y - 5$$

Replace "y" by "x"

$$f^{-1}(x) = x - 5$$

$$\boxed{f^{-1}(x) = x - 5 \; Ans.}$$

ii. Find c $f^{-1} = ?$ $Where \; f(x) = 2x - 1$

$Sol: -$

Step 1:-

Suppose $f(x) = y$

$$x = f^{-1}(y)$$

$$y = 2x - 1$$

Step 2:-

$$y = 2x - 1$$

$$y + 1 = 2x - \cancel{1} + \cancel{1}$$

$$y + 1 = 2x$$

$$x = \frac{y+1}{2}$$

Step 3:-

Replace "x" by $\boldsymbol{f^{-1}(y)}$

$$x = \frac{y+1}{2}$$

$$\boldsymbol{f^{-1}(y)} = \frac{y+1}{2}$$

Step 4:-

$$\boldsymbol{f^{-1}(y)} = \frac{y+1}{2}$$

Replace "y" by "x"

$$f^{-1}(x) = \frac{x+1}{2}$$

$$\boxed{f^{-1}(x) = \frac{x+1}{2}\ Ans.}$$

- **Types of Functions :-**

1. **Polynomial Function :-**

$p(x) = anx^{n} + an - 1\,x^{n-1} + an - 2\,x^{n-2} + - \text{-------}$

$p(x) = a_\circ$ ⟶ Constant Function

$p(x) = a_{1+\,a_2}$

$p(x) = a2x^2 + a1x + a_\circ$ ⟶ Quadratic Function

$p(x) = a3x^3 + a2x^2 + a_{1x+}a_\circ$ ⟶ Cubic Function

2. **Trigonometric Function :-**

1. $f(x) = sin\,x$
2. $f(x) = \cos x$
3. $f(x) = \tan x$
4. $f(x) = \csc x$
5. $f(x) = \sec x$
6. $f(x) = \cot x$

3. **Inverse Trigonometric Function :-**

1. $f(x) = \sin^{-1} x$
2. $f(x) = \cos^{-1} x$
3. $f(x) = \tan^{-1} x$
4. $f(x) = \csc^{-1} x$
5. $f(x) = \sec^{-1} x$

6. $f(x) = \cot^{-1} x$

4. Exponential Function :-

$f(x) = a^x \,, \qquad a \neq 1 \,, a > 0$

$f(x) = 2^x$

$f(x) = 3^x$

$f(x) = (\frac{1}{2})^x$

$f(x) = (\frac{1}{3})^x$

Common exponential function

$f(x) = e^x$ ⟶ Natural Exponential Function .

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The End of Week # 03