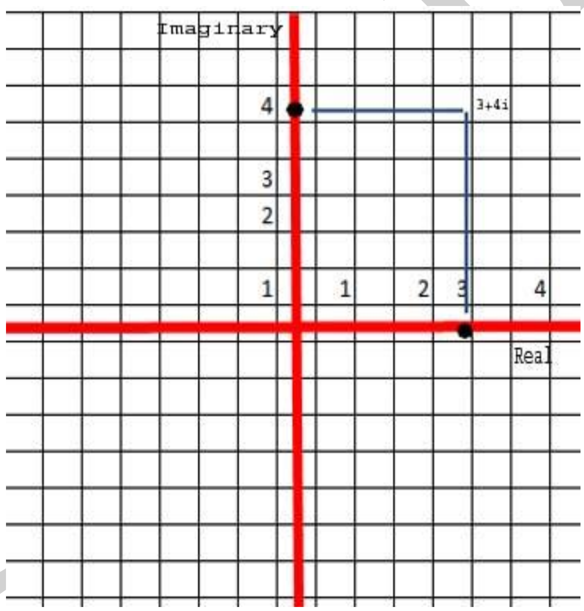
Week # 02

- **Complex Numbers in Polar Form**
- **De- Moivre's Theorem**
- **Applications of De- Moivre's Theorem**

- **Complex Numbers in Polar form :-**



1. $\cos\theta = \frac{Base}{Hyp} = \frac{x}{r} \quad => r\cos\theta = x$
2. $\sin\theta = \frac{Perp}{Hyp} = \frac{y}{r} \quad => r\sin\theta = y$
3. $Z = x + yi \quad => Z = r\cos\theta + r\sin\theta$
4. $r = \sqrt{x^2 + y^2}$
5. $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

1. Find $(x + yi)$ when its Polar Coordinate are.

(i) (1 , 30°)

$Sol{:}-$

- $r = 1 , \quad \theta = 30°$

Angles In Degree	Sin	Cos	Tan	Sec	Cosec	Cot
0°	0	1	0	1	$\frac{not}{defined}$	$\frac{not}{defined}$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	2	$\sqrt{3}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	2	$\frac{2}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$
90°	1	0	$\frac{not}{defined}$	$\frac{not}{defined}$	1	0

- $x = r\cos\theta$
- $x = (1)\cos(30°)$
- $x = \cos(30°)$

$\boxed{x = \frac{\sqrt{3}}{2}}$ ⟶ (A)

- $y = r\sin\theta$
- $y = (1)\sin(30°)$
- $y = \sin 30°$

$\boxed{y = \frac{1}{2}}$ ⟶ (B)

- $Z = x + yi$ ⟶ (1)

Put ⟶ (A) and ⟶ (B) in ⟶ (1)

- $Z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$\boxed{Z = \frac{\sqrt{3}}{2} + \frac{1}{2}i \ Ans.}$

(ii) (2 , 45°)

$Sol:-$

- $r = 2$, $\theta = 45°$
- $x = r\cos\theta$

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- $x = (2)cos(45°)$
- $x = 2\ (\frac{1}{\sqrt{2}})$

$\boxed{x = \frac{2}{\sqrt{2}}}$ ⟶ (A)

- $y = rsin\theta$
- $y = (2)sin(45°)$
- $y = 2\ \left(\frac{1}{\sqrt{2}}\ \right)$
- $y = \ (\frac{2}{\sqrt{2}}\ )$
- $y = (\frac{2}{\sqrt{2}}\ )$

$\boxed{y = \frac{2}{\sqrt{2}}}$ ⟶ (B)

- $Z = x + yi$ ⟶ (1)

Put ⟶ (A) and ⟶ (B) in ⟶ (1)

- $Z = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}\ i$

$\boxed{Z = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}\ i\ Ans.}$

(iii) (3 , −30°)

$Sol: -$

- $r = 3 \quad , \quad \theta = -30°$
- $x = rcos\theta$
- $x = (3)cos(-30°)$

- $x = (3)(-\frac{\sqrt{3}}{2})$

$\boxed{x = (3)\left(-\frac{\sqrt{3}}{2}\right)}$ ⟶ (A)

- $y = rsin\theta$
- $y = (3)sin(-30°)$
- $y = 3\left(-\frac{1}{2}\right)$

$\boxed{y = \left(-\frac{3}{2}\right)}$ ⟶ (B)

- $Z = x + yi$ ⟶ (1)

Put ⟶ (A) and ⟶ (B) in ⟶ (1)

$\boxed{Z = (3)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{3}{2}\right)i\ Ans.}$

(iv) (3 , −60°)

$Sol: -$

- $r = 3$, $\theta = -60°$
- $x = rcos$
- $x = (3)cos(-60°)$
- $x = 3\,(-\frac{1}{2})$

$\boxed{x = (-\frac{3}{2})}$ ⟶ (A)

- $y = rsin\theta$

- $y = (3)sin(-60)$

- $\boxed{y = (3)(-\frac{\sqrt{3}}{2})}$ ⟶ (B)

- $Z = x + yi$ ⟶ (1)

Put ⟶ (A) and ⟶ (B) in ⟶ (1)

- $Z = -\frac{3}{2} + (3)(-\frac{\sqrt{3}}{2})\ i$

$$\boxed{Z = -\frac{3}{2} + (3)(-\frac{\sqrt{3}}{2})\ i\ Ans.}$$

2. Find the cordinates of the following Complex Numbers.

(i) $(1, 1)$, (x, y)

$Sol:-$

- $x = 1, y = 1$

We know that
$r = \sqrt{x^2 + y^2}$

- $r = \sqrt{1^2 + 1^1}$

- $r = \sqrt{1 + 1}$

- $r = \sqrt{2}$

- $\boxed{r = \sqrt{2}}$

Now we know that $\theta = tan^{-1}\left(\frac{y}{x}\right)$

- $\theta = tan^{-1}\left(\frac{\cancel{1}}{\cancel{1}}\right)$

- $\theta = tan^{-1}(1)$
- $\theta = 45° \ or \ \frac{\pi}{4}$

$$\boxed{\theta = 45° \quad or \quad \frac{\pi}{4} \quad Ans.}$$

(ii) $(\frac{\sqrt{3}}{2}, \frac{1}{2})$, (x, y)

$Sol:-$

- $x = \frac{\sqrt{3}}{2}, y = \frac{1}{2}$

We know that
$r = \sqrt{x^2+y^2}$

- $r = \sqrt{(\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2}$ => $r = \sqrt{\left(\frac{(\sqrt{3})^2}{(2)^2} + \left(\frac{1}{2}\right)^2\right)}$
- $r = \sqrt{(\frac{3}{4}) + (\frac{1}{4})}$
- $r = \sqrt{\frac{3+1}{4}}$
- $r = \left(\sqrt{\frac{4}{4}}\right)$
- $\boxed{r = 1}$

Now we know that $\theta = tan^{-1}\left(\frac{y}{x}\right)$

- $\theta = tan^{-1}\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$

- $\theta = tan^{-1}(\frac{1}{\sqrt{3}})$

- $\theta = 30°$ or $\frac{\pi}{6}$

- $\boxed{\theta = 30° \quad or \quad \frac{\pi}{6} \quad Ans.}$

(iii) $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$, (x, y)

$Sol: -$

- $x = \frac{\sqrt{3}}{2}$, $y = -\frac{1}{2}$

We know that
$r = \sqrt{x^2+y^2}$

- $r = \sqrt{(\frac{\sqrt{3}}{2})^2 + (-\frac{1}{2})^2}$ => $\sqrt{\left(\frac{(\sqrt{3})^{2}}{(2)^2} + \left(-\frac{1}{2}\right)^2\right)}$

- $r = \sqrt{(\frac{3}{4}) + (\frac{1}{4})}$

- $r = \sqrt{\frac{3+1}{4}}$

- $r = \left(\sqrt{\frac{4}{4}}\right)$

- $\boxed{r = 1}$

Now we know that $\theta = tan^{-1}\left(\frac{y}{x}\right)$

- $\theta = tan^{-1}\left(\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$

- $\theta = tan^{-1}(-\frac{1}{\sqrt{3}})$

- $\theta = -30°$ or $-\frac{\pi}{6}$

$$\boxed{\theta = -30° \quad or \quad -\frac{\pi}{6} \quad Ans.}$$

(iv) $(1, -1)$, (x, y)

$Sol{:}-$

- $x = 1, y = -1$

We know that
$r = \sqrt{x^2+y^2}$

- $r = \sqrt{1^2 + (-1)^2}$
- $r = \sqrt{1+1}$
- $r = \sqrt{2}$
- $\boxed{r = \sqrt{2}}$

Now we know that $\theta = tan^{-1}\left(\frac{y}{x}\right)$

- $\theta = tan^{-1}\left(\frac{-1}{1}\right)$
- $\theta = tan^{-1}(-1)$
- $\theta = -45°\ or -\frac{\pi}{4}$

$$\boxed{\theta = -45° \quad or \quad -\frac{\pi}{4} \quad Ans.}$$

- **De- Moivre's Theorem :-**

- $Z = x + yi$
- $Z = rcos\theta + rsin\theta i$

I. $x = rcos\theta$
II. $y = rsin\theta$
III. $r = \sqrt{x^2+y^2}$
IV. $\theta = tan^{-1}\left(\frac{y}{x}\right)$

- $Z = \not{Z}$

- $(x + yi)^n = (r\cos\theta + ir\sin\theta)^n$

Taking " r^n " Common

- $(x + yi)^n = r^n(\cos\theta + i\sin\theta)$

Convert Power into Angel

- $(x + yi)^n = r^n(\cos(\mathrm{n})(\theta) + i\sin(n)(\theta)$

- $\boxed{(x + yi)^n = r^n(\cos \mathrm{n}\theta + i\sin n\theta)}$

- Find the Real and Imaginary parts of the following Complex Numbers using De- Moivre's Theorem.

(1) $(\sqrt{3} + i)^2$

Sol: $-$
we know that

$Z = \not{Z}$

- $(x + yi)^n = r^n(\cos(\mathrm{n})(\theta) + i\sin(n)(\theta)$

- $(\sqrt{3} + i)^2 = r^2(\cos 2\theta + i\sin 2\theta)$ ⟶ (1)

Find $R = ?$, $\theta = ?$

We know that

- $r = \sqrt{x^2 + y^2}$

- $r = \sqrt{(\sqrt{3})^{2} + 1^{2}}$

- $r = \sqrt{3 + 1}$

- $r = \sqrt{4}$

$\boxed{r = 2}$ $\longrightarrow$ (A)

Now we also know that $\theta = tan^{-1}\left(\frac{y}{x}\right)$

- $\theta = tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
- $\theta = 30°$
- $\boxed{\theta = 30°}$ $\longrightarrow$ (B)

Put $\longrightarrow$ (A) and $\longrightarrow$ (B) in $\longrightarrow$ (1)

$(\sqrt{3} + i)^2 = r^2(cos2\theta + isin2\theta)$

- $(\sqrt{3} + i)^2 = (2)^2(cos2(30°) + isin2(30°))$
- $(\sqrt{3} + i)^2 = 4(cos60° + isin60°)$
- $(\sqrt{3} + i)^2 = 4(\frac{1}{2} + i\frac{\sqrt{3}}{2})$
- $(\sqrt{3} + i)^2 = \frac{\cancel{4}}{\cancel{2}} + \cancel{4}\frac{\sqrt{3}}{\cancel{2}}i$
- $(\sqrt{3} + i)^2 = 2 + 2\sqrt{3i}$

$\boxed{Real\ Part = 2\ Ans.}$

$\boxed{\text{Imaginary Part} = 2\sqrt{3i}\ Ans.}$

(2) $(-\frac{1}{2}\ \ ,-\frac{\sqrt{3}}{2}i)^3$

$Sol: -$

$we\ know\ that$

$Z = \cancel{Z}$

- $(x + yi)^n = r^n(\cos(n)(\theta) + isin(n)(\theta)$
- $(-\frac{1}{2} + (-\frac{\sqrt{3}}{2}i))^3 = r^3\ (cos3(\theta) + i\ sin3(\theta))$

- $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)\right)^3 = r^3\,(cos3(\theta) + i\,sin3(\theta))$ ⟶ (1)

$Find \quad R = ? \quad , \quad \theta = ?$

We know that

- $r = \sqrt{x^2+y^2}$

- $r = \sqrt{(-\frac{1}{2})^2 + \frac{(\sqrt{3})^2}{(-2)^2}}$

- $r = \sqrt{\frac{1}{4}+\frac{3}{4}}$

- $r = \sqrt{\frac{3+1}{4}}$

- $r = \left(\sqrt{\frac{4}{4}}\right)$

- $\boxed{r = 1}$ ⟶ (A)

Now we also know that $\theta = tan^{-1}\left(\frac{y}{x}\right)$

- $\theta = tan^{-1}\left(\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{-2}}\right)$

- $\theta = tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right)$

- $\theta = tan^{-1}(\sqrt{3})$

- $\theta = 60°$

- $\boxed{\theta = 60°}$ ⟶ (B)

Put ⟶ (A) and ⟶ (B) in ⟶ (1)

- $(-\frac{1}{2}-\frac{\sqrt{3}}{2}i))^3 = r^3\ (cos3(\theta) + i\ sin3(\theta))$

- $(-\frac{1}{2}-\frac{\sqrt{3}}{2}i))^3 = (1)^3(cos3(60°) + isin3(60°)$

- $(-\frac{1}{2}-\frac{\sqrt{3}}{2}i))^3 = 1 + cos180° + isin180°$

- $(-\frac{1}{2}-\frac{\sqrt{3}}{2}i))^3 = 1(0 + i1)$

- $(-\frac{1}{2}-\frac{\sqrt{3}}{2}i))^3 = 1(i1)$

- $(-\frac{1}{2}-\frac{\sqrt{3}}{2}i))^3 = 0 + 1i$

$\boxed{Real\ Part = 0\ Ans.}$

$\boxed{Imaginary\ Part = 1i\ Ans.}$

(iii) $(\ 1 , -\sqrt{3i}\ )^5$

$Sol: -$
$we\ know\ that$

$Z = \not{Z}$

- $(x + yi)^n\ = r^n(\cos(\text{n})(\theta) + isin(n)(\theta)$

- $(1 + (-\sqrt{3}i))^5\ = r^5(cos5(\theta) + isin5(\theta))$

- $(1 - \sqrt{3}i)^5 = r^5(\ cos5\theta + isin5\theta\ ) \longrightarrow$ (1)

We know that

- $r = \sqrt{x^2+y^2}$

- $r = \sqrt{(1)^2 + (-\sqrt{3})^2}$

- $r = \sqrt{1 + (-\sqrt{3})^2}$
- $r = \sqrt{1+3}$
- $r = \sqrt{4}$
- $\boxed{r = 2}$ ⟶ (A)

Now we also know that $\theta = tan^{-1}\left(\frac{y}{x}\right)$

- $\theta = tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)$
- $\theta = \tan^{-1}(-\sqrt{3})$
- $\boxed{\theta = -60°}$ ⟶ (B)
- Put ⟶ (A) and ⟶ (B) in ⟶ (1)
- $(1-\sqrt{3}i)^5 = r^5(\ cos5\theta + isin5\theta\ )$
- $(1-\sqrt{3}i)^5 = 2^5(cos5(-60° + isin5(-60°)\ )$
- $(1-\sqrt{3}i)^5 = 32\ (\cos\ (-300°) + isin(-300°)$
- $(1-\sqrt{3}i)^5 = 32(\frac{1}{2} - \frac{\sqrt{3}}{2}i)$
- $(1-\sqrt{3}i)^5 = \left(\frac{32}{2}\right) - 32\ \frac{\sqrt{3}}{2}i$
- $(1-\sqrt{3}i)^5 = 16 - 16\sqrt{3i}$

$\boxed{Real\ Part = 16\ Ans.}$

$\boxed{Imaginary\ Part = -16\sqrt{3}i\ Ans.}$

(v) $(\sqrt{3}\,,\ +\ i)^3$

$Sol:-$
$we\ know\ that$

$Z = \not{Z}$

- $(x+yi)^n = r^n(\cos(\mathrm{n})(\theta) + isin(n)(\theta)$

- $(\sqrt{3}+i)^3 = r^3(cos3(\theta) + isin3(\theta))$ ⟶ (1)

We know that

- $r = \sqrt{x^2 + y^2}$
-
- $r = \sqrt{(\sqrt{3})^2 + (1)^2}$

- $r = \sqrt{3+1}$

- $r = \sqrt{4}$

$\boxed{r = 2}$ ⟶ (A)

Now we also know that $\theta = tan^{-1}\left(\frac{y}{x}\right)$

- $\theta = tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

- $\boxed{\theta = 30°}$ ⟶ (B)

- Put ⟶ (A) and ⟶ (B) in ⟶ (1)

- $(\sqrt{3}+i)^3 = r^3(cos3(\theta) + isin3(\theta))$

- $(\sqrt{3}+i)^3 \ = 2^3(cos3(30°) + isin3(30°))$
- $(\sqrt{3}+i)^3 \ = 8(cos90° + isin90°)$
- $(\sqrt{3}+i)^3 \ = 8(0+1i)$
- $(\sqrt{3}+i)^3 \ = 0+8i$

$\boxed{Real\ Part\ = 0\ Ans.}$

$\boxed{Imaginary\ Part = 8i\ Ans.}$

- **Solved by Formulas:-**

(i) $(\sqrt{3}+i)^3$

$Sol:-$

$we\ know\ that$

$\boxed{(a+b)^3 = a^3+b^3+3a^2b+3ab^2}$

- $(\sqrt{3}+i)^3 = (\sqrt{3})^3 + (i)^3 + 3(\sqrt{3})^2\,(i) + 3(\sqrt{3})\,(i)^2$
- $(\sqrt{3}+i)^3 = (\sqrt{3})^2.(\sqrt{3}) + (-i) + 9i + 3\sqrt{3}\,(-1)$
- $(\sqrt{3}+i)^3 = 3\sqrt{3} - i + 9i - 3\sqrt{3}$
- $(\sqrt{3}+i)^3 = 0+8i$

$\boxed{Real\ Part = 0\ Ans.}$

$\boxed{Imaginary\ Part = 8i\ Ans.}$

(ii) $(\sqrt{3}+i)^2$

$Sol:-$
$We\ know\ that$

$\boxed{(a+b)^2 = a^2+b^2+2ab}$

Formulas

$\boxed{(a+b)^3 = a^3+b^3+3a^2b+3ab^2}$

$\boxed{(a+b)^2 = a^2+b^2+2ab}$

$\boxed{(a-b)^3 = a^3-b^3-3a^2b+3ab^2}$

$\boxed{a^2-b^2 = (a+b)(a-b)}$

$\boxed{a^3-b^3 = (a-b)(a^2+ab+b^2}$

$\boxed{i^3 = -i}$

- $(\sqrt{3} + i)^2 = (\sqrt{3})^2 + (i)^2 + 2(\sqrt{3})(i)$
- $(\sqrt{3} + i)^2 = 3 - 1 + 2\sqrt{3}i$
- $(\sqrt{3} + i)^2 = 2 + 2\sqrt{3}i$

$\boxed{Real\ Part = 2\ Ans.}$

$\boxed{Imaginary\ Part = 2\sqrt{3}i\ Ans.}$

(iii) $(1 + i)^3$

$Sol: -$
$we\ know\ that$

$\boxed{(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2}$

- $(1 + i)^3 = (1)^3 + (i)^3 + 3(1)^2(i) + 3(1)(i)^2 \quad \because \quad i^3 = -i$
- $(1 + i)^3 = 1 + (-i) + 3i + 3(-1)$
- $(1 + i)^3 = 1 - i + 3i - 3$
- $(1 + i)^3 = 1 - 3 - i + 3i$
- $(1 + i)^3 = -2 + 2i$

$\boxed{Real\ Part = -2\ Ans.}$

$\boxed{Imaginary\ Part = +2i\ \ Ans.}$

(iv) $(2 + 3i)^2$

$Sol: -$

$We\ know\ that$

$\boxed{(a+b)^2 = a^2 + b^2 + 2ab}$

- $(2 + 3i)^2 = (2)^2 + (3i)^2 + 2(2)(3i)$
- $(2 + 3i)^2 = 4 + 9(i)^2 + 12i$
- $(2 + 3i)^2 = 4 + 9(-1) + 12i$
- $(2 + 3i)^2 = 4 - 9 + 12i$
- $(2 + 3i)^2 = -5 + 12i$

$\boxed{Real\ Part = -5\ Ans.}$

$\boxed{Imaginary\ Part = +12i\ \ Ans.}$

(v) $(3 - 5i)^3$

Sol:−
we know that

$\boxed{(a - b)^3 = a^{3-}b^3 - 3a^2b + 3ab^2}$

- $(3 - 5i)^3 = (3)^3 - (5i)^3 - 3(3)^2(5i) + 3(3)(5i)^2$ $\qquad \because i^2 = -1$
- $(3 - 5i)^3 = 27 - 125(i)^3 - 3(45i) + 3(75)(i)^2$ $\qquad \because i^3 = -i$
- $(3 - 5i)^3 = 27 - 125(-i) - 135i + 225(-1)$
- $(3 - 5i)^3 = 27 - 225 + 125i - 135i$
- $(3 - 5i)^3 = -198 - 10i$

$\boxed{Real\ Part = -198\ Ans.}$

$\boxed{Imaginary\ Part = -10i\ \ Ans.}$

The End of Week # 02