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Subject : Multivariate Calculus

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Function: Every Input which exact only one output is called Function.

For example:

$D = \{1, 2, 3\}$

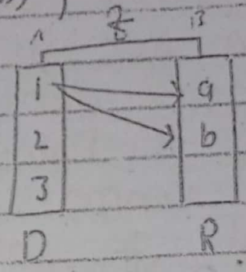
$\times$

$R = \{a, b\}$

$D \times R = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

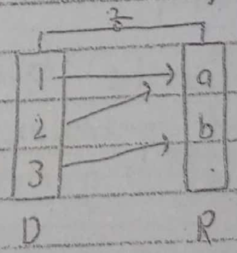
Binary Relation:

$R_1 = \{(1,a), (1,b)\}$



Not function.

$R_2 = \{(1,a), (2,a), (3,b)\}$



function.

x	-3	-2	-1	0	1	2	3
$y = f(x) = x$	-3	-2	-1	0	1	2	3

x = Domain, Input, Independent variable, pre-Image.

$f(x)$ = Range, Dependent variable, output, Image, Co-domain f

x	f(x)
0	0
-1	-1
-2	-2
-3	-3
1	1
2	2
3	3

function of one variable.

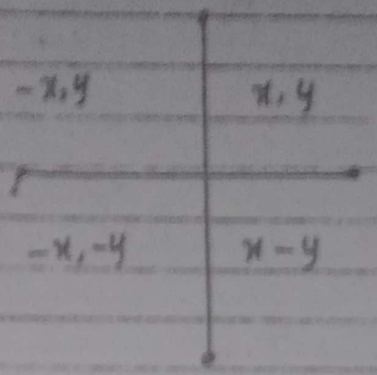
Function of two Variable.

Definition: ~~A~~ function of two variable is a rule that assigns to each ordered pair of real number (x,y) in a set of D a unique real number denoted by f (x,y) The set D is Domain of f and its range is that set of values that f takes on that

$$\{ f(x,y) / (x,y) \in D \}$$

we often write $z = f(x,y)$ to make explicit - that value taken on by f at general point (x,y)

The variable x & y are independent variable and z is dependent variable.



Note:

$(x, y) \in \mathbb{R}$

$f(x) = \frac{1}{x} + \frac{1}{y}$

$x \neq 0$ & $y \neq 0$

(I) $z = x^2 + y^2$

$z = (x, y) = x + y$

$z = f(x, y) = \frac{1}{x} + \frac{1}{y}, \; x \neq 0, \; y \neq 0$

$z = f(x, y) = e^{x+y} + \sin x$

$z = f(x, y) = \sin^{-1} x + \cos^{-1} y$

Example # 01 For each of the following function evaluate $f(3, 2)$

(a) $f(x, y) = \sqrt{\frac{x+y+1}{x-1}}$

(b) $f(x, y) = x \ln(y^2 - x)$

(a) $f(x, y) = \sqrt{\frac{x+y+1}{x-1}}$

Sol:-

$f(3, 2) = \sqrt{\frac{3+2+1}{3-1}}$

$f(3, 2) = \sqrt{\frac{6}{2}}$

(b) $f(x,y) = x\ln(y^2 - x)$

Sol:-

$f(3,2) = 3\ln((2)^2 - 3)$

$= 3\ln(4-3)$

$= 3\ln(1)$

$= 3(0)$

$\boxed{f(3,2) = 0}$

Note

$\ln(1) = 0$

Note

$f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$

Domain:-

$x \neq 1$

$x+y+1 \geq 0$

$Dom = \{x,y / x+y+1 \geq 0$

limit function of two variable:-

Definition: Let f be a function of two variables whose Domain D includes points arbitrarily close to (a,b) then we say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L and we write

$f(x,y) = L$ unique, Define.

$$\lim_{(x,y)\to(a,b)}$$

OR

$$\lim_{\substack{x\to a \\ y\to b}} f(x,y) = L$$

Note.

$$\lim_{x\to a} f(x) = L$$

Rule of the limit of the function of two variables

(a) Sum Rule

$$\lim_{(x,y)\to(a,b)} \big(f(x,y) + g(x,y)\big) = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)$$

(2) Difference Rule

$$\lim_{(x,y)\to(a,b)} \big(f(x,y) - g(x,y)\big) = \lim_{(x,y)\to(a,b)} f(x,y) - \lim_{(x,y)\to(a,b)} g(x,y)$$

(3) Product Rule

$$\lim_{(x,y)\to(a,b)} \big(f(x,y) \cdot g(x,y)\big) = \lim_{(x,y)\to(a,b)} f(x,y) \cdot \lim_{(x,y)\to(a,b)} g(x,y)$$

(4) Quotient Rule

$$\lim_{(x,y)\to(a,b)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$$

(5) Power Rule

$$\lim_{(x,y)\to(a,b)} (f(x,y))^n = \left(\lim_{(x,y)\to(a,b)} f(x,y)\right)^n$$

—— x —— x —— x —— x —— x —— x ——

Find the limit if it exist.

Q # 1

$$\lim_{(x,y)\to(1,2)} (5x^3 - x^2y^2)$$

Sol:- put the value.

$$\lim_{(x,y)\to(1,2)} (5x^3 - x^2y^2)$$

$$\lim_{(x,y)\to(1,2)} 5x^3 - \lim_{(x,y)\to(1,2)} x^2y^2$$

put value.

$$5(1)^3 - (1)^2(2)^2$$

$$5 - (1)(4)$$

$$5 - 4$$

(2

$\boxed{1}$ Ans

(2) $\lim_{(x,y)\to(1,-1)} e^{-xy} \cos(x+y)$

Sol:

$\lim_{(x,y)\to(1,-1)} e^{-xy} \cos(x+y)$

$e^{-(1)(-1)} \cos(1+-1)$

$e^{1} \cos(0)$ Note $\cos 0 = 1$

$e(1)$

$\boxed{e}$ Ans

Q

Q # 3 $\lim_{(x,y)\to(2,1)} \frac{4-xy}{x^2+3y^2}$

Sol:- using Quotient Rule.

$$\frac{\lim_{(x,y)\to(2,1)} 4-xy}{\lim_{(x,y)\to(2,1)} x^2+3y^2}$$

$$\frac{4-(2)(1)}{(2)^2+3(1)^2}$$

$= \frac{4-2}{} = \frac{2}{}$

Q # 4 $\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right)$

Sol:- $\ln\left(\frac{\lim_{(x,y)\to(1,0)} 1+y^2}{\lim_{(x,y)\to(1,0)} x^2+xy}\right)$

$\ln\left(\frac{1+(0)^2}{(1)^2+(1)(0)}\right)$

$\ln\left(\frac{1}{1}\right)$

$\ln(1) = \boxed{0}$ Ans

Q # 05 $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$

$\left(\frac{0}{0}\right)$ (indeterminate Form)

Sol: $\lim_{(x,y)\to(0,0)} \frac{(x^2)^2-(y^2)^2}{x^2+y^2}$

$\lim_{(x,y)\to(0,0)} \frac{\cancel{(x^2+y^2)}(x^2-y^2)}{\cancel{x^2+y^2}}$

$\lim_{(x,y)\to(0,0)} x^2-y^2$

0^2-0^2

$\boxed{0}$ Ans

$$\lim_{(x,y)\to(9,9)} \frac{x-y}{\sqrt{x}-\sqrt{y}} \to \text{H.W}$$

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Q#6 $$\lim_{(x,y)\to(1,1)} \frac{x^3-y^3}{x-y}$$

Sol:

$$\lim_{(x,y)\to(1,1)} \frac{\cancel{(x-y)}(x^2+xy+y^2)}{\cancel{(x-y)}}$$

$$\lim_{(x,y)\to(1,1)} (x^2+xy+y^2)$$

$$(1)^2+(1)(1)+(1)^2$$

$$=(1+1+1)$$

$$=\boxed{3} \text{ ans}$$

Q# 7 $$\lim_{(x,y)\to(4,4)} \frac{\sqrt{x}-\sqrt{y}}{(x-y)}$$

Sol:

$$\lim_{(x,y)\to(4,4)} \frac{\sqrt{x}-\sqrt{y}}{(x-y)}$$

Take conjugation.

$$= \frac{\sqrt{x}-\sqrt{y}}{(x-y)} \times \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

$$= \frac{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}{(x-y)(\sqrt{x}+\sqrt{y})}$$

$$\frac{(\sqrt{x})^2 - (\sqrt{y})^2}{(x-y)(\sqrt{x}+\sqrt{y})}$$

$$\frac{\cancel{(x-y)}}{\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}$$

$$\lim_{(x,y)\to(4,4)} \frac{1}{\sqrt{x}+\sqrt{y}}$$

$$= \frac{1}{\sqrt{4}+\sqrt{4}}$$

$$= \frac{1}{2+2} = \boxed{1/4}$$

$f(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \to \text{First principle Rule.}$$

or

$$f'(x) \quad \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

w.r.t y

$$f(x)\ (x,y) = \lim_{\Delta x \to 0,\, y} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}.$$

$$\frac{\partial f}{\partial f} = f_y(x,y) = \lim_{x, \Delta y \to 0} \frac{f(x, y+\Delta x) - f}{\Delta y}$$

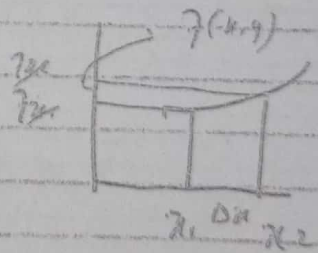
$\boxed{\text{We can use } h \text{ instead of } \Delta x \text{ is}}$

If $f(x,y)$ is a function of two variables its partial derivative are the functions f_x and f_y defined by.

$$f_{(x)}(x,y) = \lim_{h \to 0} \frac{f(x+h, y+h) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

$$f_x(x,y,z) = \lim_{h \to 0} \frac{f(x+h, y, z)}{h}$$



15 — 38 Find the first partial Derivative of the function.

$\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

15). $z(x,y) = y^5 - 3xy$

Sol:

$z(x,y) = y^5 - 3xy$

Diff partially w.r.t x

$\frac{\partial z}{\partial x} = z(x) = \frac{\partial}{\partial x}(y^5 - 3xy)$

$= \frac{\partial y^5}{\partial x} - \frac{\partial}{\partial x} 3xy$

$= 0 - 3y \frac{\partial}{\partial x} x$

$= 0 - 3y(1)$

$\boxed{\frac{\partial z}{\partial x} = -3y}$

$z(x, y) = y^5 - 3xy$

Diff partially w.r.t y

$$\frac{\partial}{\partial y} z(x, y) = \frac{\partial}{\partial y}(y^5 - 3xy)$$

$$= \frac{\partial}{\partial y}(y^5) - \frac{\partial (3xy)}{\partial y}$$

$$\boxed{\frac{\partial z}{\partial y} = 5y^4 - 3x}$$

(12) $z(x, t) = e^{-t} \cos \pi x$

Sol:

Diff partially w.r.t t

$$\frac{\partial z}{\partial t} = z_t = \frac{\partial}{\partial t}\left(e^{-t} \cos \pi x\right)$$

$$= \cos \pi x \frac{\partial}{\partial t}\left(e^{-t}\right)$$

$$= \cos \pi x \, e^{-t} \cdot \frac{\partial}{\partial t}(-t)$$

$$= \cos \pi x \, e^{-t} (-1)$$

$$\boxed{\frac{\partial z}{\partial t} = -e^{-t} \cos \pi x}$$

$$z(x,t) = e^{-t} \cos \pi x$$

$$\frac{\partial z}{\partial x} = \frac{\partial (e^{-t} \cos \pi x)}{\partial x}$$

$$= e^{-t} \frac{\partial}{\partial x} (\cos \pi x)$$

Note: $\frac{d}{dx} \cos x = -\sin x$

$$= e^{-t} (-\sin \pi x) \frac{\partial}{\partial x} (\pi x)$$

$$\boxed{\frac{\partial z}{\partial x} = -\pi e^{-t} \sin \pi x}$$

(9) $z = (2x + 3y)^{10}$

$$\frac{\partial z}{\partial x} = \frac{\partial (2x + 3y)^{10}}{\partial x}$$ Extended power Rule.

$$= 10 (2x + 3y)^9 \frac{\partial (2x + 3y)}{\partial x}$$

$$= 10 (2x + 3y)^9 \frac{\partial (2x)}{\partial x} + \frac{\partial (3y)}{\partial x}$$

$$= 10 (2x + 3y)^9 \, 2 + 0$$

$$\boxed{\frac{\partial y}{\partial x} = 20 (2x + 3y)^9}$$

20) $z = \tan xy$

Sol:

$$\frac{\partial y}{\partial x} = \frac{\partial(\tan xy)}{\partial x}$$

Note $\frac{d}{dx}\tan x = \sec^2$

$$= \sec^2 xy \frac{\partial(xy)}{\partial x}$$

$$= \sec^2 xy\,(y)$$

$$\boxed{\frac{\partial y}{\partial x} = y\sec^2 xy}$$

21) $z(x,y) = \frac{x-y}{(x+y)}$

Sol: using Quotient Rule.

$$= \frac{(x+y)\frac{\partial}{\partial x}(x-y) - (x-y)\frac{\partial}{\partial x}(x+y)}{(x+y)^2}$$

$$= \frac{(x+y)(1) - (x-y)(1+0)}{(x+y)^2}$$

$$= \frac{(x+y) - (x-y)}{(x+y)^2}$$

$$= \frac{\cancel{x}+y-\cancel{x}+y}{(x+y)^2} = \boxed{\frac{2y}{(x+y)^2}}$$

(18) $z(x,t) = \sqrt{x}\ \ln t$

Sol: Diff w.r.t x.

$$= \frac{\partial(\sqrt{x}\ \ln t)}{\partial x}$$

$$= \ln t \frac{\partial}{\partial x}\sqrt{x}$$

$$= \ln t \frac{\partial (x)^{1/2}}{\partial x}$$ power Rule Apply

$$= \ln t\ \tfrac{1}{2}(x)^{1/2-1}$$

$$= \ln t\ \tfrac{1}{2}x^{-1/2}$$

Note. $x^{-1/2} = \frac{1}{\sqrt{x}}$

$$= \ln t\ \tfrac{1}{2}x^{-1/2}$$

$$= \ln t \frac{1}{2\sqrt{x}}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{\ln t}{2\sqrt{x}}}$$

$f(x,t) \quad \sqrt{x} \ln t$

Sol:

Diff w.r.t t.

$$= \frac{\partial(\sqrt{x}\ln t)}{\partial(x)}$$

$$= \sqrt{x} \frac{\partial \ln t}{\partial \cancel{x} t}$$

Note:
$\frac{d}{dt} \ln t = 1/t$

$$= \sqrt{x} \; 1/t$$

$$\boxed{\frac{\partial f}{\partial t} = \frac{\sqrt{x}}{t}}$$

22) $f(x,y) = x^y$

Sol: Diff w.r.t x.

$$\frac{\partial f}{\partial x} = \frac{\partial(x^y)}{\partial x}$$

$$= y x^{y-1}$$

$$\boxed{\frac{\partial f}{\partial x} = y x^{y-1}}$$

$f(x, y) = x^y$

Sol: Diff w.r.t y

$$\frac{\partial f}{\partial y} = \frac{\partial x^y}{\partial y}$$

Note: $\frac{d}{dy} x^y = x^y \ln x$

x^2 = Exponential
2^x = Quadratic

$$\boxed{\frac{\partial f}{\partial y} = x^y \ln x}$$

(23) $w(\alpha, \beta) = \sin\alpha \cos\beta$

Sol: w.r.t α

$$\frac{\partial w}{\partial \alpha}(\alpha, \beta) = \frac{\partial(\sin\alpha \cos\beta)}{\partial \alpha}$$

$$= \frac{\partial(\sin\alpha \cos\beta)}{\partial \alpha}$$

$$= \cos\beta \frac{\partial \sin\alpha}{\partial \alpha}$$

$$\boxed{\frac{\partial w}{\partial \alpha} = \cos\beta \cos\alpha}$$

$\cos\beta \sin\alpha$

$\omega(\alpha, \beta) = \sin\alpha \cos\beta$

Sol: Diff w.r.t β

$$\frac{\partial(\alpha, \beta)}{\partial\beta} = \frac{\partial(\sin\alpha \cos\beta}{\partial\beta}$$

$$= \frac{\sin\alpha(-\sin\beta)}{\partial\beta}$$

$$\boxed{\frac{\partial\omega}{\partial\beta} = -\sin\alpha \sin\beta}$$

24) $\omega(u, v) = \frac{e^v}{u+v^2}$

Sol: Diff w.r.t u

using Quotient Rule.

$$\frac{\partial(u,v)}{\partial(u)} = \frac{\partial\left(\frac{e^v}{u+v^2}\right)}{\partial u}$$

$$= \frac{(u+v^2)\frac{\partial}{\partial u}e^v - e^v\frac{\partial(u+v^2)}{\partial u}}{(u+v^2)^2}$$

$$= \frac{(u+v^2)e^v(0) - e^v(1+0)}{}$$

$$= \frac{e^v(u+v^2)(0) - 1e^v}{}$$

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Q9) $w(u,v) = \dfrac{e^{v}}{u+v^2}$

Sol: Diff w.r.t v

Quotient Rule.

$$\frac{\partial w}{\partial v} = \frac{\partial\left(\dfrac{e^{v}}{u+v^2}\right)}{\partial v}$$

$$= \frac{(u+v^2)\dfrac{\partial(e^{v})}{\partial v} - (e^{v})\dfrac{\partial(u+v^2)}{\partial v}}{(u+v^2)^2}$$

$$= \frac{(u+v^2)\, e^{v}\ln e - e^{v}(0+2v)}{(u+v^2)^2}$$

$$= \frac{(u+v^2)\, e^{v}\ln e - e^{v}(2v)}{(u+v^2)^2}$$

$$= \frac{(u+v^2)\, e^{v}\ln e - 2v e^{v}}{(u+v^2)^2}$$

$$\boxed{\frac{\partial w}{\partial v} = \frac{e^{v}\ln e\,(u+v^2) - 2v e^{v}}{(u+v^2)^2}}$$

25) $z(r, s) = r \ln(r^2+s^2)$

Sol: Diff w.r.t r

$$\frac{\partial z}{\partial r} = \frac{\partial (r \ln(r^2+s^2))}{\partial r}$$

$$= r \frac{\partial (\ln(r^2+s^2))}{\partial r} + \left(\ln(r^2+s^2)\right) \frac{\partial r}{\partial r}$$

$$= r\left(\frac{1}{r^2+s^2} \frac{\partial (r^2+s^2)}{\partial r} + \ln(r^2+s^2)\right)$$

$$= r\left(\frac{1}{r^2+s^2}(2r+0) + \ln(r^2+s^2)\right)$$

$$= r\left(\frac{2r}{r^2+s^2} + \ln(r^2+s^2)\right)$$

$$\boxed{\frac{\partial z}{\partial r} = \frac{2r^2}{r^2+s^2} + r\ln(r^2+s^2)}$$

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$z(r,s) = r\ln(r^2+s^2)$

Sol.

w.r.t s

$$\frac{\partial z}{\partial s} = \frac{\partial}{\partial s}\left(r\ln(r^2+s^2)\right)$$

$$= r\frac{\partial \ln(r^2+s^2)}{\partial s} + \ln(r^2+s^2)\frac{d}{ds}(r)$$

$$= r\left(\frac{1}{r^2+s^2}\frac{\partial(r^2+s^2)}{\partial s} + 0\right)$$

$$= r\left(\frac{1}{r^2+s^2}(0+2s)\right)$$

$$= r\frac{(2s)}{r^2+s^2}$$

$$\boxed{\frac{\partial z}{\partial s} = \frac{2rs}{r^2+s^2}}$$

26) $f(x,y,z) = xyz^2 \tan y$

Sol:

$$\frac{\partial f}{\partial x} = \frac{\partial (xyz^2 \tan y)}{\partial x}$$

$$= yz^2 \tan y \frac{\partial x}{\partial x}$$

$$\boxed{f_x = yz^2 \tan y}$$

w.r.t y

$$f_y = xyz^2 \tan y$$

$$\frac{\partial y}{\partial y} = \frac{\partial (xyz^2 \tan y)}{\partial y}$$

$$= xz^2 \left(y \frac{\partial}{\partial y} (\tan y) + (\tan y) \frac{\partial}{\partial y}(y) \right)$$

$$= xz^2 (y \sec^2 y + \tan y \cdot (1))$$

$$\boxed{f_y = xz^2 (y \sec^2 y + \tan y)}$$

w.r.t z

$$\boxed{f_z = 2xyz \tan y}$$

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27) $z(x, y, z, t) = \dfrac{xy^2}{t+2z}$

Sol:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{xy^2}{t+2z}\right)$$

$$= \left(\frac{y^2}{t+2z}\right)\frac{\partial (x)}{\partial x}$$

$$\boxed{z(x) = \frac{y^2}{t+2z}} \longrightarrow (i)$$

w.r.t y

$$\boxed{\text{(ii)} \quad z_y = \frac{2xy}{t+2z}} \longrightarrow (ii)$$

$$\frac{\partial z}{\partial z} = \frac{\partial}{\partial z}\left(\frac{xy^2}{t+2z}\right)$$

$$= \frac{\partial}{\partial z}(xy^2)(t+2z)^{-1}$$

$$= \frac{\partial}{\partial z}(xy^2)(t+2z)^{-1}$$

$$= xy^2 \frac{\partial}{\partial z}(t+2z)^{-1}$$

$$= xy^2(-1)(t+2z)^{-1-1} \frac{\partial}{\partial z}(t+2z)$$

$$= \frac{-xy^2}{(t+2z)^2}(0+2)$$

$$\boxed{\frac{\partial z}{\partial z} = -\frac{2xy^2}{(t+2z)^2}}$$

Definition:

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{z(x+\Delta x, y) - z(x,y)}{\Delta x}$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{z(x, y+\Delta x) - z(x,y)}{\Delta y}$$

Use Definition by partial differentiation

Find $\frac{\partial z}{\partial x}$

Sol:

$$z(x,y) = x^2+y^2 \rightarrow (1)$$

$$z(x+\Delta x, y) = (x+\Delta x)^2 + y^2 \rightarrow (ii)$$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{z(x+\Delta x, y) - z(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{(x+\Delta x)^2 + y^2 - (x^2+y^2)}{\Delta x}\right]$$

$$\lim_{\Delta x \to 0} \left[\frac{\cancel{x^2} + 2\Delta x x + \Delta x^2 + \cancel{y^2} - \cancel{x^2} - \cancel{y^2}}{\Delta x}\right]$$

$$\lim \left[\frac{\Delta x(2x+\Delta x)}{}\right]$$

$$= 2x + 0$$

$$\boxed{\frac{\partial z}{\partial x} = 2x}$$

(ii) $z(x,y) = x + y^2$

$$\frac{\partial z}{\partial y} = ?$$

Sol:

$$\frac{\partial z}{\partial y}$$

$$z(x,y) = x + y^2 \rightarrow (i)$$

$$z(x, y+\Delta y) = x + (y+\Delta y)^2 \rightarrow (ii)$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0}\left[\frac{z(x, y+\Delta y) - z(x,y)}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0}\left[\frac{x + (y+\Delta y)^2 - (x+y^2)}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0}\left[\frac{\cancel{x} + \cancel{y^2} + 2y\Delta y + \Delta y^2 - \cancel{x} - \cancel{y^2}}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0}\left[\frac{\cancel{\Delta y}(2y + \Delta y)}{\cancel{\Delta y}}\right]$$

$$= 2y + 0$$

$$\boxed{\frac{\partial z}{\partial y} = 2y}$$

③ $f(x,y) = x^3 + y^3$

④ $f(x,y) = x^3 y^3$

(2) $\quad z(x,y) = x^2 y$

Sol: $\quad \partial z / \partial x = ?$

$z(x,y) = x^2 y \rightarrow ①$

$z(x+\Delta x, y) = (x+\Delta x)^2 y \rightarrow ⓘⓘ$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \left[\frac{z(x+\Delta x, y) - z(x,y)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{(x+\Delta x)^2 y - (x^2 y)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{(x^2 + 2x\Delta x + \Delta x^2) y - x^2 y}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\cancel{x^2 y} + 2xy\Delta x + \Delta x^2 y - \cancel{x^2 y}}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\cancel{\Delta x}(2xy + \Delta x y)}{\cancel{\Delta x}} \right]$$

$$= \left[2xy + 0 \right]$$

$$\boxed{\frac{\partial z}{\partial x} = 2xy}$$

w.r.t y

$\partial z \quad 2$

3) $f(x,y) = x^3 + y^2$

$\frac{\partial f}{\partial x} = ?$

Sol:

$f(x,y) = x^3 + y^2 \rightarrow (i)$

$f(x+\Delta x, y) = (x+\Delta x)^3 + y^2 \rightarrow (ii)$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{(x+\Delta x)^3 + y^2 - (x^3 + y^2)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\cancel{x^3} + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + \cancel{y^2} - \cancel{x^3} - \cancel{y^2}}{\Delta x} \right]$$

$$\lim_{\Delta x \to 0} \left[\frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3}{\Delta x} \right]$$

$$\lim_{\Delta x \to 0} \left[\frac{\cancel{\Delta x}(3x^2 + 3x\Delta x + \Delta x^2)}{\cancel{\Delta x}} \right]$$

$$\left[3x^2 + 0 + 0 \right]$$

$$\lim_{\Delta x \to 0} = 3x^2$$

$$\boxed{\frac{\partial f}{\partial x} = 3x^2}$$

w.r.t x

$$\boxed{\frac{\partial f}{\partial x} = 2xy}$$

ans

Higher Derivatives:

if f is a function of two variables then its partial derivatives f_x and f_y are also function of two variable, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$ $(f_y)_x$, and $(f_y)_y$ which are called the second partial derivatives of f.

if $z = f(x, y)$ we use the following notation.

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x}$$

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Q # 07 $f(x,y) = x^3 + x^2y^3 - 2y^2$

Find:

i) F_{xx}

ii) f_{yx}

iii) f_{xy}

iv) f_{yy}

Sol:

$f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_x = 3x^2 + 2xy^3 - 0$

$f_x = 3x^2 + 2xy^3$

again diff partially w.r.t x

$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3)$

$\boxed{f_{xx} = 6x + 2y^3}$

$f_x = 3x^2 + 2xy^3$

Sol: Diff partially w.r.t y

$f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2y^3)$

$= 0 + 6xy^2$

$\boxed{f_{xy} = 6xy^2}$

$f(x,y) = x^3 + x^2y^3 - 2y^3$

Sol:

$f_y = 0 + 3x^2y^2 - 6y^2$

$$\boxed{f_y = 3x^2y^2 - 6y^2}$$

$$\boxed{f_{yy} = 6x^2y - 12y}$$

again Diff partially w.r.t ~~y~~ x

$f_{yx} = 6xy^2 - 0$

$$\boxed{f_{yx} = 6xy^2}$$

Note

$f_{xy} = f_{yx}$

Q # 02 $f(x,y,z) = xz - 5x^2y^3z^4$

Find:

1) $f_{xx} = ?$
2) $f_{xy} = ?$
3) $f_{yx} = ?$
4) $f_{xz} = ?$
5) $f_{xyz} = ?$

5^2

Sol:

① $f_{xx} = ?$

$f(x,y,z) = xz - 5x^2y^3z^4$

$$f(x) = \frac{\partial(xz - 5x^2y^3z^4)}{\partial x}$$

$f_x = z - 10xy^3z^4$

again Diff partially w.r.t x.

$$f_{xx} = \frac{\partial(z - 10xy^3z^4)}{\partial x}$$

$= 0 - 10y^3z^4$

$$\boxed{f_{xx} = -10y^3z^4}$$

2) $f_{xy} = ?$

$f_x = z - 10xy^3z^4$

Diff w.r.t y

$$f_{xy} = \frac{\partial(z - 10xy^3z^4)}{\partial y}$$

$= 0 - 30xy^2z^4$

$$\boxed{f_{xy} = -30xy^2z^4}$$

(3) $f_{yx} = ?$

$$f(x,y,z) = xz - 5x^2y^3z^4$$

$$f_y = \frac{\partial (xz - 5x^2y^3z^4)}{\partial y}$$

$$= 0 - 15x^2y^2z^4$$

$$\boxed{f_y = -15x^2y^2z^4}$$

$$f_{yx} = \frac{\partial}{\partial x}(-15x^2y^2z^4)$$

$$\boxed{f_{yx} = -30xy^2z^4}$$

4) $f_{xz} = ?$

$$f_x = z - 10xy^3z^4$$

$$f_{xz} = \frac{\partial (z - 10xy^3z^4)}{\partial z}$$

$$= 1 - 40xy^3z^3$$

$$\boxed{f_{xz} = 1 - 40xy^3z^3}$$

① $f(x,y) = e^{-x}\sin(x+y)$

② $f(x,y) = y\sin xy$

③ $f(x,y) = e^{xy}\ln y$

$f(x,y) = \sin^2(x-3y)$

④ $f(x,y) = \cos^2(3x-y^2)$

⑤ $f(x,y,z) = \ln(x+2y$

⑥ $f(x,y,z) = \sin^{-1}(xyz)$

⑦ $f(x,y,z) = \sec^{-1}(x+y+z)$

⑧ $f(x,y,z) = e^{-(x^2+y^2+z^2)}$

f_{xy}, f_{yx}

f_{xx}, f_{yy}

(5) $f_{xyz} = ?$

$$f_{xy} = -30xy^2z^4$$

$$f_{xyz} = \frac{\partial(-30xy^2z^4)}{\partial z}$$

$$= -120xy^2z^3$$

$$\boxed{f_{xyz} = -120xy^2z^3}$$

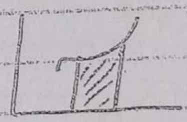
week # 02, 03

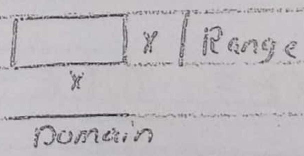
Double Integral

Triple Integral

Integration: Find area under the curve.

$f(x) = x^2 \quad -1 \le x \le 2$

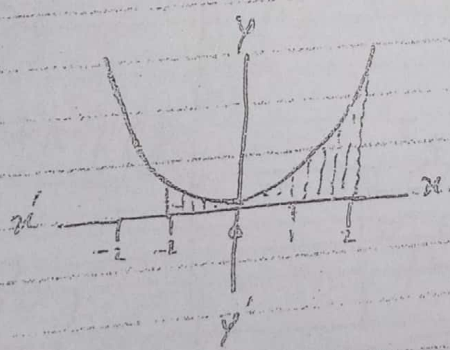




Area = L × w

= x × x

Area = x^2



$$\int_{-1}^{2} x^2 \, dx$$

$$= \left. \frac{x^{2+1}}{2+1} \right|_{-1}^{2} = \left. \frac{x^3}{3} \right|_{-1}^{2} = \left(\frac{(2)^3}{3} - \frac{(-1)^3}{3} \right)$$

$$= \left(\frac{8}{3} + \frac{1}{3} \right)$$

$$= \left(\frac{9}{3} \right)$$

$$\boxed{\text{Area} = 3}$$

$$\frac{\Delta y}{\Delta x} = \text{Slope} = \frac{\text{Rise}}{\text{Run}}$$

Double Integration :-

$$\int_a^b \int_c^d f(x,y)\, dx\, dy = \int_c^d \int_a^b f(x,y)\, dy\, dx$$

$$\int_a^b \left[\int_c^d f(x,y)\, dx \right] dy = \int_c^d \left[\int_a^b f(x,y)\, dy \right] dx.$$

Example # 01

a) $\int_0^3 \left[\int_1^2 x^2 y\, dy \right] dx$ w.r.t y

Sol :-

$$= \int_1^2 x^2 y\, dy$$

$$= \left. \frac{x^2 y^2}{2} \right|_1^2$$

$$= \frac{x^2}{2} \left[(2)^2 - (1)^2 \right]$$

$$= \frac{x^2}{2} (4 - 1)$$

$$= \frac{3x^2}{2}$$

$$= \int_0^3 \frac{3x^2}{2}\, dx$$

$$= \frac{3}{2}\int_0^3 x^2\,dx$$

$$= \frac{\cancel{3}}{2}\,\frac{x^3}{\cancel{3}}\Bigg|_0^3$$

$$= \frac{1}{2}\left[3^3 - 0^3\right] = \frac{27}{2} \text{ Ans.}$$

(b) $\int_1^2\int_0^3 x^2 y\,dx\,dy = \frac{27}{2}$ ans

if f is continuous on the rectangle

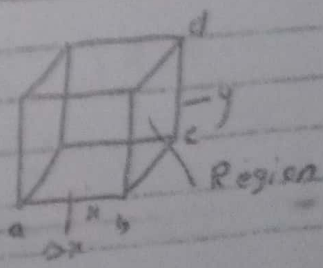
$R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$ then

$$\iint_R f(x,y)\,\underbrace{dA}_{\text{area}} = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

Note

$dA = dx\,dy$

$dA = dy\,dx$



Example # 02

Evaluate the double integral

$$\iint_R (x - 3y^2)\, dA \quad \text{where}$$

$$R = \{(x,y) \mid 0 \le x \le 2 \,/\, 1 \le y \le 2\}$$

$$\iint_R (x-3y^2)\,dA = \int_0^2\int_1^2 (x-3y^2)\,dy\,dx = \int_1^2\int_0^2 (x-3y^2)\,dx\,dy$$

$$= \int_0^2 \left[\int_1^2 (x-3y^2)\,dy\right] dx$$

$$= \int_1^2 (x-3y^2)\,dy$$

$$= \int_1^2 x\,dy - \int_1^2 3y^2\,dy$$

$$= xy\Big|_1^2 - \frac{\cancel{3}y^3}{\cancel{3}}\Big|_1^2$$

$$= xy\Big|_1^2 - y^3\Big|_1^2$$

$$= x(2-1) - ((2)^3 - (1)^3)$$

$$= x - (8-1)$$

$$= x - 7$$

$$= \int_0^2 (x-7)\, dx$$

$$= \int_0^2 x\, dx - \int_0^2 7\, dx$$

$$= \frac{x^2}{2}\Big|_0^2 - 7x\Big|_0^2$$

$$= \left(\frac{(2)^2}{2} - \frac{(0)^2}{2}\right) - 7(2-0)$$

$$= (2-0) - (14)$$

$$= \boxed{-12 \text{ Ans}}$$

Example # 03

$$\iint_R y \sin(xy)\, dA, \text{ where } R = [1,2] \times [0,\pi]$$

Sol: $\rightarrow$ $\displaystyle\int_0^\pi \int_1^2 y \sin(xy)\, dx\, dy$

$$= \int_0^\pi \left[\int_1^2 y \sin(xy)\, dx\right] dy$$

$$= \int_1^2 y \sin xy \, dx$$

$$= -\frac{\cancel{y} \cos xy}{\cancel{y}} \Big|_1^2$$

$$= -\cos 2y + \cos y$$

$$= [\cos y - \cos 2y]$$

$$= \int_0^{\pi} (\cos y - \cos 2y) \, dy$$

$$= \int_0^{\pi} \cos y \, dy - \int_0^{\pi} \cos 2y \, dy$$

$$= \sin y \Big|_0^{\pi} - \frac{\sin 2y}{2} \Big|_0^{\pi}$$

$$= \sin\pi - \sin(0) - \frac{1}{2} (\sin 2(\pi) - \sin(0))$$

$$= 0 - 0 - 0 - 0$$

$$= 0 \text{ Answer}$$

Exercise :- (7-14) calculate the iterated integrated.

(3) $\int_1^3 \int_0^1 (1 + 4xy)\, dx\, dy$

Sol :→

$$\int_1^3 \left[\int_0^1 (1 + 4xy)\, dx \right] dy$$

$$= \int_0^1 (1 + 4xy)\, dx$$

$$= \int_0^1 1\, dx + \int 4xy\, dx$$

$$= x \Big|_0^1 + \frac{4y\, x^2}{2} \Big|_0^1$$

$$= x \Big|_0^1 + 2y\, x^2 \Big|_0^1$$

$$= (1) - (0) + 2y(1)^2 - 2y(0)^2$$

$$= 1 - 0 + 2y - 0$$

$$= \boxed{1 + 2y}$$

y^2

$$= \int_1^3 (1 + 2y)\, dy$$

$$= \int_1^3 1\, dy + \int_1^3 2y\, dy$$

$$= y \Big|_1^3 + \frac{\cancel{2} y^2}{\cancel{2}} \Big|_1^3$$

$$= (3) - (1) + (3)^2 - (1)^2$$

$$= 3 - 1 + 9 - 1$$

$$= 2 + 8$$

$$\boxed{= 10} \quad \text{Ans}$$

$$\boxed{(4) \quad \cancel{\int_0^2 \int_0^{\pi/2}} \; x \cancel{\sin y}\, dy\, dx}$$

Assignment

(4) $\displaystyle \int_0^1 \int_1^2 (4x^3 - 9x^2y^2)\, dy\, dx$

Sol: →

$$\int_0^1 \left[\int_1^2 (4x^3 - 9x^2y^2)\, dy \right] dx$$

$$= \int_1^2 (4x^3 - 9x^2y^2)\, dy$$

$$= \int_1^2 4x^3\, dy - \int_1^2 9x^2y^2\, dy$$

$$= 4x^3 y \Big|_1^2 - \frac{\cancel{9}^{\,3} x^2 y^3}{\cancel{3}} \Big|_1^2$$

$$= 4x^3(2-1) - 3x^2\left((2)^3 - (1)^3\right)$$

$$= 4x^3(2-1) - 3x^2(8-1)$$

$$= 4x^3 - 3x^2(7)$$

$$= 4x^3 - 2x^2$$

$$= \int_0^1 (4x^3 - 2x^2)\, dx$$

$$= \int_0^1 4x^3\, dx - \int_0^1 2x^2\, dx$$

$$= \frac{\cancel{4}x^4}{\cancel{4}} \Big|_0^1 - \frac{2x^3}{3} \Big|_0^1$$

$$x^4 \Big|_0^1 - \frac{2x^3}{3} \Big|_0^1$$

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$= (1)^4 - (0)^4 - \frac{2(1)^3}{3} - \frac{2(0)^3}{3}$

$= 1 - 0 - 2/3 - 0$

$= 1 - 2/3$

$\boxed{= 1/3}$ Ans

Ten (5) $\int_0^2 \int_0^{\pi/2} x \sin y \, dy \, dx.$

Sol:→ $\int_0^2 \left[\int_0^{\pi/2} x \sin y \, dy \right] dx$

$= \int_0^{\pi/2} x \sin y \, dy$

$= -x \cos y \Big|_0^{\pi/2}$

$= -x \left[\cos(\pi/2) - \cos(0) \right]$

$= -x \left[0 - 1 \right] =$

$= x$

$$= \int_0^2 x \, dx$$

$$= \frac{x^2}{2}\Big]_0^2 = \frac{4-0}{2} = \boxed{2} \text{ Ans}$$

Assignment

1) $\int_0^2 \int_0^1 (2x+y)^8 \, dx \, dy$

$$= \int_0^2 \left[\int_0^1 (2x+y)^8 \, dx \right] dy$$

$$= \int_0^1 (2x+y)^8 \, dx$$

$$= \frac{(2x+y)^{8+1}}{(8+1)2}$$

$$= \frac{(2x+y)^9}{(9)2}\Big|_0^1$$

$$= \frac{(2x+y)^9}{18}\Big|_0^1$$

$$= \frac{1}{18}\left[\frac{(2x+y)^9}{18} \Big|_0^1 \right]$$

Note

$(ax+b)^n =$

$\frac{d(ax+b)^n}{dx} =$

$n(ax+b)^{n-1} \frac{d}{dx}(ax+b)$

$n(ax+b)^{n-1} a$

$an(ax+b)^{n-1}$

$\int (ax+b)^n dx =$

$\frac{(ax+b)^{n+1}}{(n+1)a}$

only for when the power of x 1

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$$= \frac{1}{18}\left[(2(1)+y)^9 - (2(0)+y)^9\right]$$

$$= \frac{1}{18}\left[\int (2+y)^9 - (y)^9\right]$$

$$= \frac{1}{18}\left[\int_0^2 (2+y)^9 dy - \int_0^2 y^9 \, dy\right]$$

$$= \frac{1}{18}\left[\frac{(2+y)^{10}}{10}\Big|_0^2 - \frac{y^{10}}{10}\Big|_0^2\right].$$

$$= \frac{1}{18}\left[\frac{(2+2)^{10}}{10} - \frac{(2+0)^{10}}{10}\right] - \left[\frac{(2)^{10}}{10} - \frac{(0)^{10}}{10}\right]$$

$$= \frac{1}{18}\left[\frac{(4)^{10}}{10} - \frac{(2)^{10}}{10}\right] - \left[\frac{(2)^{10}}{10}\right]$$

$$= \frac{1}{18}\left[\frac{(4)^{10} - (2)^{10}}{10}\right] - \left[\frac{(2)^{10}}{10}\right]$$

$$= \frac{1}{18}\left[\frac{1048576 - 1024}{10}\right] - \left[\frac{1024}{10}\right]$$

$$= \frac{1}{18}\left[\frac{1047552}{10} - \frac{1024}{10}\right]$$

$$\frac{1}{18}\left[\frac{1047552 - 1024}{10}\right]$$

$$\frac{1}{18}\left[\frac{1046528}{10}\right]$$

$$\frac{1}{18}\left[\frac{104652.8}{1}\right]$$

$$\boxed{5814.0444}$$ ans

(15-22) Calculate the double Integral

15) $\iint_R (6x^2y^3 - 5y^4)\, dA \quad R = \{(x,y) / 0 \le x \le 3, 0 \le y \le 1\}$

Sol:→

$$\int_0^3 \int_0^1 (6x^2y^3 - 5y^4)\, dy\, dx$$

$$= \int_0^3 \left[\int_0^1 (6x^2y^3 - 5y^4)\, dy\right] dx$$

$$= \int_0^3 (6x^2y^3 - 5y^4)\, dy$$

$$= \int_0^1 6x^2y^3\, dy - \int_0^1 5y^4\, dy$$

$$= 6x^2 \frac{y^{3+1}}{3+1}\Big|_0^1 - 5\frac{y^{4+1}}{4+1}\Big|_0^1$$

$$= \overset{3}{\cancel{6}}x^2 \frac{y^4}{\cancel{4}_2}\Big|_0^1 - \cancel{5}\frac{y^5}{\cancel{5}}\Big|_0^1$$

$$= \frac{3x^2 y^4}{2}\Big|_0^1 - y^5\Big|_0^1$$

$$= \frac{3x^2}{2}\left[(1)^4 - (0)\right] - \left[(1)^5 - (0)^5\right]$$

$$= \frac{3x^2}{2}\left[1 - 0\right] - \left[1\right]$$

$$= \frac{3x^2}{2} - 1$$

$$= \frac{3x^2 - 2}{2}$$

$$= \int_0^3 \frac{3x^2 - 2}{2}$$

$$= \frac{1}{2} \int_0^3 (3x^2 - 2)\, dx$$

$$= \frac{1}{2} \int_0^3 3x^2\, dx - \int 2\, dx$$

$$= \frac{1}{2} \times \frac{3x^{2+1}}{2+1} \Big|_0^3 - 2x \Big|_0^3$$

$$= \frac{1}{2} \left[\frac{\cancel{3}x^3}{\cancel{3}} \Big|_0^3 - 2x \Big|_0^3 \right]$$

$$= \frac{1}{2} \left[x^3 \Big|_0^3 - 2x \Big|_0^3 \right]$$

$$= \frac{1}{2} \left[(3)^3 - (0)^3 \right] - 2 \left[3 - 0 \right]$$

$$= \frac{1}{2} \left[27 - 0 \right] - 6 \left. \right]$$

$$= \frac{1}{2} \left[27 - 6 \right]$$

$$= \frac{21}{2}$$

$\boxed{10.5}$ Ans

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(b) $\iint_R \cos(x+2y)\, dA$ $R = \{(x,y) \mid 0 \le x \le \pi,\ 0 \le y \le \pi/2\}$

$$\int_0^{\pi}\int_0^{\pi/2} \cos(x+2y)\, dy\, dx$$

Sol: →

$$\int_0^{\pi/2} \cos(x+2y)\, dy$$

$$= \left.\frac{\sin(x+2y)}{2}\right|_0^{\pi/2}$$

$$= \frac{\sin(x+2(\pi/2))}{2} - \frac{\sin(x+2(0))}{2}$$

$$= \frac{\sin(x+\pi)}{2} - \frac{\sin(x)}{2}$$

$$= \frac{1}{2}\left[\sin(x+\pi) - \sin x\right]$$

$$= \frac{1}{2}\int_0^{\pi} \sin(x+\pi)\, dx - \frac{1}{2}\int_0^{\pi} \sin x\, dx$$

$$= -\frac{1}{2}\cos(x+\pi)\Big|_0^{\pi} + \frac{1}{2}\cos x\Big|_0^{\pi}$$

$$= -\frac{1}{2}\left[\cos(\pi+\pi) - \cos(0+\pi)\right] + \frac{1}{2}\left[\cos\pi - \cos(0)\right]$$

$$= -\frac{1}{2}\left[\cos 2\pi - \cos(\pi)\right] + \frac{1}{2}\left(-1-1\right)$$

$$= -\frac{1}{2}\left[1 - (-1)\right] + \frac{1}{2}\left[-2\right]$$

$$= -\frac{1}{2}\left[1+1\right] + \frac{1}{2}\left[-2\right]$$

$$= -\frac{1}{\cancel{2}}\left[\cancel{2}\right] + \frac{1}{\cancel{2}}\left[-\cancel{2}\right]$$

$$= -1 - 1 = \boxed{-2}$$ Ans

Q2

$$\iint_R \frac{1+x^2}{1+y^2}\, dA \quad R = \{0 \le x \le 1,\ 0 \le y \le 1\}$$

$$\int_0^1 \int_0^1 \frac{1+x^2}{1+y^2}\, dy\, dx$$

Sol $\Rightarrow$

$$\int_0^1 \left[\int_0^1 \frac{1+x^2}{1+y^2}\, dy \right] dx$$

$$= \int_0^1 \frac{1+x^2}{1+y^2}\, dy$$

$$= 1+x^2 \int_0^1 \frac{1}{1+y^2}\, dy$$

$$= (1+x^2) \int_0^1 \frac{1}{1+y^2}\, dy$$

$$= (1+x^2) \left[\tan^{-1} y \Big|_0^1 \right]$$

$$= (1+x^2) \left[\tan^{-1}(1) - \tan^{-1}(0) \right]$$

$$= (1+x^2) \left[\pi/4 - 0 \right]$$

$$= (1+x^2) \left(\frac{\pi}{4} \right)$$

CamScanner

$$\frac{\pi}{4}\left(1+x^2\right)$$

$$\frac{\pi}{4}\int_0^1 \left(1+x^2\right)\,dx$$

$$=\frac{\pi}{4}\left[x\Big|_0^1 + \frac{x^3}{3}\Big|_0^1\right]$$

$$=\frac{\pi}{4}\left[(1-0)+\frac{(1)^3}{3}-\frac{(0)^3}{3}\right]$$

$$=\frac{\pi}{4}\left[1+\frac{1}{3}\right]$$

$$=\frac{\pi}{4}\left(\frac{3+1}{3}\right)$$

$$=\frac{\pi}{\cancel{4}}\left[\frac{\cancel{4}}{3}\right]$$

$$=\pi/3 = \boxed{60^\circ}$$ Ans

Note

$$\frac{d}{dy}\tan^{-1}y=\frac{1}{1+y^2}$$

$$\int \frac{1}{(1+y^2)}\,dy=\tan^{-1}y$$

$$\tan 45^\circ = 1$$

$$\tan \pi/4 = 1$$

$$\pi/4 = \tan^{-1}(1)$$

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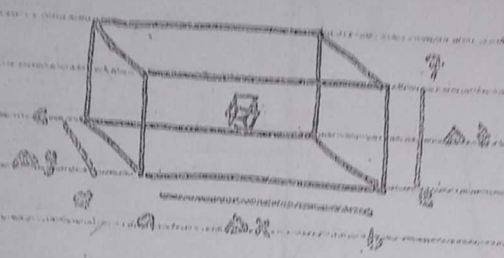
Triple Integration:-

Just we defined single integral of one variable functions and double integrals for function of two variables.

So, we can define triple Integrals for function of three variable.

$$\iiint_R f(x,y,z)\,dV = \int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$



Note
R = for Rectangular Box
V = Volume

Evaluate the triple Integral

$$\iiint_B xyz^2\,dx\,dy\,dz$$ where B is the Rectangular Box.

$$B = \{(x,y,z) / 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$$

Q: $\int_0^{4}\int_{-1}^{2}\int_0^{1} xyz^2 \, dx \, dy \, dz$

Sol:- For x

$$\int_0^1 xyz^2 \, dx$$

$$= yz^2 \int_0^1 x \, dx$$

$$= yz^2 \frac{x^2}{2} \Big|_0^1$$

$$= \frac{yz^2}{2}(1^2 - 0^2) = \boxed{\frac{yz^2}{2}}$$

For y

$$\int_{-1}^{2} \frac{yz^2}{2} \, dy$$

$$= \frac{z^2}{2} \int_{-1}^{2} y \, dy$$

$$= \frac{z^2}{2} \int_{-1}^{2} y \, dy$$

$$= \frac{z^2}{2} \times \frac{y^2}{2} \Big|_{-1}^{2}$$

$$= \frac{z^2}{4} y^2 \Big|^{2}$$

$$= \frac{z^2}{4}\left((2)^2 - (-1)^2\right)$$

Note
$(-1)^2 = 1$

$$= \frac{z^2}{4}(4-1)$$

$$= \frac{z^2}{4}(3) = \boxed{\frac{3z^2}{4}}$$

For z

$$= \int_0^3 \frac{3z^2}{4}\, dz$$

$$= \frac{3}{4}\int_0^3 z^2\, dz$$

$$= \cancel{\int}$$

$$= \frac{3}{4} \times \frac{z^{2+1}}{2+1}\Big|_0^3$$

$$= \frac{3}{4} \times \frac{z^3}{3}\Big|_0^3$$

$$= \frac{3z^3}{12}\Big|_0^3$$

$= \frac{3}{12}\left((3)^3 - (0)^3\right)$

$= \frac{\cancel{3}}{\cancel{12}}(27) = \boxed{\frac{27}{4}}$ Ans

Exercise (15-16) Page # (3.8)

Evaluate the iterated Integral

Q # 3

$$\int_0^1 \int_0^z \int_0^{x+z} (6xz)\, dy\, dx\, dz$$

Sol: For y

$$= \int_0^{x+z} (6xz)\, dy$$

$$= 6xz \int_0^{x+z} 1\, dy$$

$$= 6xzy \Big|_0^{x+z}$$

$$= 6xz(x+z-0)$$

$$= 6xz(x+z)$$

$$= \boxed{6x^2z + 6xz^2}$$

For x

$$= \int_0^z (6x^2 z + 6x z^2)\, dx$$

$$= \int_0^z 6x^2 z \, dx + \int_0^z 6x z^2 \, dx$$

$$= 6z \int_0^z x^2 \, dx + 6z^2 \int_0^z x \, dx$$

$$= 6z \frac{x^{2+1}}{2+1}\Big|_0^z + 6z^2 \frac{x^{1+1}}{1+1}\Big|_0^z$$

$$= 6z \frac{x^3}{3}\Big|_0^z + 6z^2 \frac{x^2}{2}\Big|_0^z$$

$$= 2zx^3\Big|_0^z + 3z^2x^2\Big|_0^z$$

$$= 2z(z^3 - (0)^3) + 3z^2((z)^2 - (0)^2)$$

$$= 2z(z^3) + 3z^2(z^2)$$

$$= 2z^4 + 3z^4$$

$$= \boxed{5z^4}$$

For z

$$\int_0^1 5z^4\,dz$$

$$= 5\int_0^1 z^4\,dz$$

$$= 5\frac{z^{4+1}}{4+1}\Bigg|_0^1$$

$$= \cancel{5}\frac{z^5}{\cancel{5}}\Bigg|_0^1$$

$$= z^5\Big|_0^1$$

$$= (1)^5 - (0)^5$$

$$= \boxed{1}$$ Ans

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(4)
$$\int_0^1 \int_x^{2x} \int_0^y 2xyz \, dz \, dy \, dx$$

Sol:- For z

$$= \int_0^y 2xyz \, dz$$

$$= 2xy \int_0^y z \, dz$$

$$= 2xy \frac{z^{1+1}}{1+1} \Big|_0^y$$

$$= \frac{\cancel{2}xyz^2}{\cancel{2}} \Big|_0^y$$

$$= xyz^2 \Big|_0^y$$

$$= xy(z^2) \Big|_0^y$$

$$= xy(y^2 - (0)^2)$$

$$= xy(y^2)$$

$= \boxed{x y^3}$

= <u>For y</u>

$$= \int_{x}^{2x} x y^3 \, dy$$

$$= x \int_{x}^{2x} y^3 \, dy$$

$$= x \left. \frac{y^{3+1}}{3+1} \right|_{x}^{2x}$$

$$= x \left. \frac{y^4}{4} \right|_{x}^{2x}$$

$$= \frac{x}{4} \left. \left(y^4 \right) \right|_{x}^{2x}$$

$$= \frac{x}{4} \left((2x)^4 - (x)^4 \right)$$

$$= \frac{x}{4} \left(16x^4 - x^4 \right)$$

$$= \frac{x}{4} \left(15x^4 \right) = \boxed{\frac{15x^5}{4}}$$

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For x

$$= \int_0^1 \frac{15x^5}{4}\, dx$$

$$= \frac{15}{4} \int_0^1 x^5\, dx$$

$$= \frac{15}{4} \left. \frac{x^{5+1}}{5+1} \right|_0^1$$

$$= \frac{15}{4} \times \frac{x^6}{6}$$

$$= \left. \frac{15}{24} x^6 \right|_0^1$$

$$= \frac{15}{24} \left((1)^6 - (0)^6 \right)$$

$$= \frac{15}{24} (1 - 0)$$

$$= \frac{15}{24} (1)$$

$$= \boxed{\frac{15}{24}} \text{ Ans}$$

Q # 5

$$\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} z e^y \, dx \, dz \, dy$$

Sol :→

For x

$$= \int_0^{\sqrt{1-z^2}} z e^y \, dx$$

$$= z e^y \int_0^{\sqrt{1-z^2}} 1 \, dx$$

$$= z e^y x \Big|_0^{\sqrt{1-z^2}}$$

$$= z e^y \left[\left(\sqrt{1-z^2}\right) - (0) \right]$$

$$= \boxed{z e^y \left(\sqrt{1-z^2} \right)}$$

Note

$\frac{d}{dx} x = 1$

$\int 1 \, dx = x$

For z

$$\int_0^1 z e^y \left(\sqrt{1-z^2} \right) dz$$

$$= e^{y}\int_0^1 (z)(1-z^2)^{1/2}\,dz$$

xing + ÷ by -2

$$= \frac{-e^{y}}{2}\int_0^1 (1-z^2)^{1/2}(-2z)\,dz$$

$$= -\frac{e^{y}}{2}\left[\frac{(1-z^2)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Big|_0^1\right]$$

$$= -\frac{e^{y}}{\cancel{2}}\left[\frac{(1-z^2)^{3/2}}{3/\cancel{2}}\Big|_0^1\right]$$

$$= -e^{y}\left[\frac{(1-z^2)^{3/2}}{3}\Big|_0^1\right]$$

$$= \frac{-e^{y}}{3}\left[(1-z^2)^{3/2}\Big|_0^1\right]$$

$$= \frac{-e^{y}}{3}\left[\left((1-(1)^2\right)^{3/2} - (1-0^2)^{3/2}\right]$$

$$= \frac{-e^{y}}{3}\left[(1-1)^{3/2} - (1)^{3/2}\right]$$

$$= \frac{-e^{y}}{3}[0 - 1]$$

Note

$\sqrt{\ } = 1/2$

$$\int (\sin x)^5 \cos x\,dx$$

$$\frac{\sin x^{5+1}}{5+1}$$

$$= \frac{\sin x^6}{6}$$

$$-\frac{e^y}{3}(-1)$$

$$= \boxed{\frac{e^y}{3}}$$

written by Ahmad Ali

1/12/2018

For y

$$= \int_0^3 \frac{e^y}{3}\, dy$$

$$= \frac{1}{3}\int_0^3 e^y\, dy$$

$$= \frac{1}{3} e^y \int_0^3 dy$$

$$= \frac{e^y}{3}\int_0^3 1\, dy$$

$$= \frac{e^y}{3} y \Big|_0^3$$

$$= e^y \Big|_0^3$$

$$= e^3 - e^0$$

$$= e^3 - 1$$

Week # 4

Line - Integral and surface Integral.

Definition: Let D be a set in R^2 (a plane region).
A vector field on R^2 is a function F that assigns to each point (x,y) in D a two dimensional vector $F(x,y)$

$$F(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$$

Note:
that P and Q are scalar function of two variables

Definition: Let E be a subset of R^3.
A vector field on R^3 is a function F that assigns to each point (x,y,z) in E

three dimensional vector ~~field~~ $F(x,y,z)$

$$F(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$$

Gradient Fields:

if f is a scalar function of two variables

represented by ∇ (Neble) mean operator

$$\nabla = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right)$$ For two dimensional

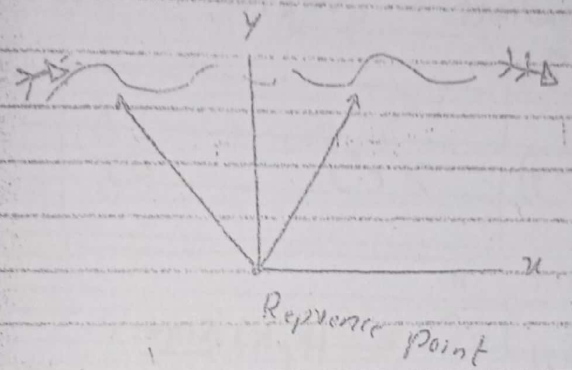
$$\nabla = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$ For three dimensional.

$$\nabla F = (x, y) = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j}$$

$$\nabla F = (x, y, z) = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Note
∇ = Scalar convert to vector.

operator = work to change occur.



Q # 1 Find the gradiant vector field of.

$$z(x, y) = x^2 y - y^3$$

Diff w.r.t x

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 y - y^3)$$

$$= 2xy - 0$$

$$\boxed{\frac{\partial z}{\partial x} = 2xy \longrightarrow (i)}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}(x^2 y - y^3)$$

$$\boxed{\frac{\partial f}{\partial y} = x^2 - 3y^2}$$

$$\nabla = \Delta f(x,y) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$\nabla f(x,y) = (2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$$

Find the gradient vector field of f:

Q # 2 $f(x,y)$ xe^{xy}

Diff w.r.t x.

$$\frac{\partial f}{\partial x} = \frac{\partial (xe^{xy})}{\partial x}$$

$$= x\frac{\partial}{\partial x}e^{xy} + e^{xy}\frac{\partial x}{\partial x}$$

$$= xe^{xy}\frac{d}{\partial x}xy + e^{xy}.1$$

$$= xe^{xy}(1.y) + e^{xy}$$

$$= xye^{xy} + e^{xy}$$

$$\boxed{\frac{\partial f}{\partial x} = e^{xy}(xy+1)} \longrightarrow 1$$

Note.
using product formula.

$$\frac{d}{dx}(a.b) =$$

$$a\frac{d}{dx}b + b\frac{d}{dx}a$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} xe^{xy} \quad \text{w.r.t } y$$

$$= x \frac{\partial}{\partial y} e^{xy} + e^{xy} \frac{\partial x}{\partial y}$$

$$= x e^{xy} \frac{\partial}{\partial y} xy + e^{xy} \frac{\partial x}{\partial y}$$

$$= x e^{xy} x \cdot 1 + e^{xy} (0)$$

$$= x^2 e^{xy} + 0$$

$$\boxed{\frac{\partial z}{\partial y} = x^2 e^{xy}} \rightarrow (2)$$

$$\nabla z(x,y) = z_x \hat{i} + z_y \hat{j}$$

$$\boxed{\nabla z(x,y) = e^{xy}(xy+1)\hat{i} + (x^2 e^{xy})\hat{j}}$$

OR

$$\nabla z(x,y) = e^{xy}\left[(xy+1)\hat{i} + (x^2)\hat{j}\right] \quad \underline{\text{Ans}}$$

Q # 3 $f(x,y) = \tan(3x - 4y)$

Sol:- w.r.t x

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\tan(3x-4y)$$

$$= \sec^2(3x-4y)\,\frac{\partial}{\partial x}(3x-4y)$$

$$= \sec^2(3x-4y) \cdot 3 - 0$$

$$\boxed{\frac{\partial f}{\partial x} = 3\sec^2(3x-4y)} \longrightarrow (i)$$

w.r.t y

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(\tan(3x-4y)$$

$$= \sec^2(3x-4y)\,\frac{\partial}{\partial y}(3x-4y)$$

$$= \sec^2(3x-4y)\; 0 - 4$$

$$\boxed{\frac{\partial f}{\partial y} = -4\sec^2(3x-4y)} \longrightarrow (ii)$$

$$\nabla f(x,y) = f_x \hat{i} + f_y \hat{j}$$

$$\boxed{\nabla f(x,y) = 3\sec^2(3x-4y)\,\hat{i} - 4\sec^2(3x-4y)\,\hat{j}}$$

Answer

OR

$$\boxed{\nabla f(x,y) = \sec^2(3x-4y)\,(3i-4j)}$$ Ans

Q # 4: $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

Sol:

w.r.t x.

$$f(x,y,z) = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}$$

$$= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}$$

$$= \frac{1}{2}(x^2+y^2+z^2)^{1/2-1}\frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x + 0 + 0$$

$$= \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}}$$

$$\boxed{\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}} \longrightarrow (i)$$

w.r.t y.

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\sqrt{x^2+y^2+z^2}$$

$$= \frac{\partial}{\partial y}(x^2+y^2+z^2)^{1/2}$$

$$= \frac{\partial}{\partial y}$$

$$= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\frac{\partial}{\partial y}(x^2+y^2+z^2)$$

$$= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(0+2y+0)$$

$$= \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2}(\cancel{2}y)$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}} \longrightarrow (2)$$

w.r.t z.

$$\frac{\partial z}{\partial z} = \frac{\partial}{\partial z}\sqrt{x^2+y^2+z^2}$$

$$= \frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2}$$

$$= \frac{1}{2}(x^2+y^2+z^2)^{1/2-1}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(0+0+2z)$$

$$\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2}(\cancel{2}z)$$

$$\boxed{\frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}} \longrightarrow (3)$$

$$\nabla f(x,y,z) = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$= \left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\hat{i} + \left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\hat{j} + \left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\hat{k}$$

$$\boxed{\nabla f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}$$

Ans

Q#5 $z(x,y,z) = x\cos(y/z)$

Sol:. w.r.t x

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} x\cos(y/z)$$

$$\frac{\partial z}{\partial x} = 1 \cdot \cos(y/z)$$

$$\boxed{\frac{\partial z}{\partial x} = \cos(y/z)} \longrightarrow (1)$$

$z(x,y,z) = x\cos(y/z)$

Diff w.r.t y

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} x\cos(y/z)$$

$$= x(-\sin(y/z)) \frac{\partial}{\partial y}(y/z)$$

$$= -x\sin(y/z)\ \frac{1}{z} \cdot 1$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-x\sin(y/z)}{z}} \longrightarrow (2)$$

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$$f(x, y, z) = x \cos(y/z)$$

Diff w.r.t z

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z} x \cos(y/z)$$

$$= x(-\sin(y/z)) \frac{\partial}{\partial z}(y/z)$$

$$= -x \sin(y/z) \frac{\partial}{\partial z}(y \cdot z^{-1})$$

$$= -x \sin(y/z) \cdot y \; -z^{-2}$$

$$= x \sin(y/z)\, y\, z^{-2}$$

$$= \frac{xy}{z^2} \sin(y/z)$$

$$\boxed{\frac{\partial f}{\partial z} = \frac{xy}{z^2} \sin(y/z)} \longrightarrow (3)$$

$$\nabla f(x, y, z) = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

$$= \cos(y/z)\hat{i} - \frac{x}{z} \sin(y/z)\hat{j} + \frac{xy}{z^2} \sin(y/z)$$

$$\boxed{\nabla f(x,y,z) = \cos(y/z)\hat{i} - \frac{x}{z} \sin(y/z)\hat{j} + \frac{xy}{z^2} \sin(y/z)\hat{k}}$$

Answer