(BS. [illegible] CS) 1st Semester

Discrete Structures

Lecture No.1 Logic

Course Objective:

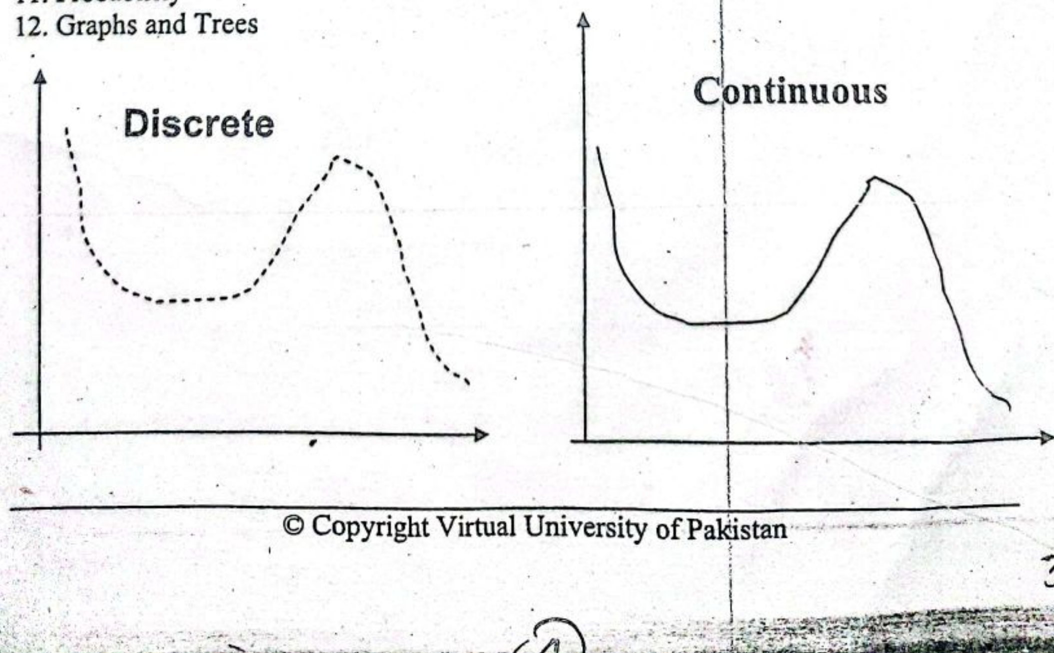
1.Express statements with the precision of formal logic
2.Analyze arguments to test their validity
3.Apply the basic properties and operations related to sets
4.Apply to sets the basic properties and operations related to relations and functions
5.Define terms recursively
6.Prove a formula using mathematical induction
7.Prove statements using direct and indirect methods
8.Compute probability of simple and conditional events
9.Identify and use the formulas of combinatorics in different problems
10.Illustrate the basic definitions of graph theory and properties of graphs
11.Relate each major topic in Discrete Mathematics to an application area in computing

1.Recommended Books:

1.Discrete Mathematics with Applications (second edition) by Susanna S. Epp
2.Discrete Mathematics and Its Applications (fourth edition) by Kenneth H. Rosen
1.Discrete Mathematics by Ross and Wright

MAIN TOPICS:

1. Logic
2. Sets & Operations on sets
3. Relations & Their Properties
4. Functions
5. Sequences & Series
6. Recurrence Relations
7. Mathematical Induction
8. Loop Invariants
9. Loop Invariants
10. Combinatorics
11. Probability
12. Graphs and Trees



Set of Integers:

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3 -2 -1 0 1 2

Set of Real Numbers:

-3 -2 -1 0 1 2 3

What is Discrete Mathematics?

Discrete Mathematics concerns processes that consist of a sequence of individual steps.

LOGIC:

Logic is the study of the principles and methods that distinguish between a valid and an invalid argument.

SIMPLE STATEMENT:

A statement is a declarative sentence that is either true or false but not both.
A statement is also referred to as a **proposition**

EXAMPLES:

a. 2+2 = 4,
b. It is Sunday today

If a proposition is true, we say that it has a **truth value** of "**true**".
If a proposition is false, its truth value is "**false**".
The truth values "**true**" and "**false**" are, respectively, denoted by the letters **T** and **F**.

EXAMPLES:

Propositions

1) Grass is green.
2) $4+2=6$
3) $4+2=7$
4) There are four fingers in a hand.

Not Propositions

1) Close the door.
2) x is greater than 2.
3) He is very rich

Rule:

If the sentence is preceded by other sentences that make the pronoun or variable reference clear, then the sentence is a statement.

Example:

$x = 1$
$x > 2$
"$x > 2$" is a statement with truth-value FALSE.

Example

Bill Gates is an American
He is very rich
"He is very rich" is a statement with truth-value TRUE.

UNDERSTANDING STATEMENTS

1) $x + 2$ is positive. — Not a statement
2) May I come in? — Not a statement
3) Logic is interesting. — A statement
4) It is hot today. — A statement
5) $-1 > 0$ — A statement
6) $x + y = 12$ — Not a statement

COMPOUND STATEMENT:

Simple statements could be used to build a compound statement.

LOGICAL CONNECTIVES

EXAMPLES:

1. "3 + 2 = 5" **and** "Lahore is a city in Pakistan"
2. "The grass is green" **or** " It is hot today"
3. "Discrete Mathematics is **not** difficult to me"

AND, OR, NOT are called LOGICAL CONNECTIVES.

SYMBOLIC REPRESENTATION

Statements are symbolically represented by letters such as ***p, q, r,...***

EXAMPLES:

p = "Islamabad is the capital of Pakistan"
q = "17 is divisible by 3"

CONNECTIVE	MEANINGS	SYMBOLS	CALLED
Negation	not	$\sim$	Tilde
Conjunction	and	$\wedge$	Hat
Disjunction	or	$\vee$	Vel
Conditional	if…then…	$\rightarrow$	Arrow
Biconditional	if and only if	$\leftrightarrow$	Double arrow

EXAMPLES

p = "Islamabad is the capital of Pakistan"
q = "17 is divisible by 3"
$p \wedge q$ = "Islamabad is the capital of Pakistan and 17 is divisible by 3"
$p \vee q$ = "Islamabad is the capital of Pakistan or 17 is divisible by 3"
$\sim p$ = "It is not the case that Islamabad is the capital of Pakistan"
or simply "Islamabad is not the capital of Pakistan"

TRANSLATING FROM ENGLISH TO SYMBOLS

Let p = "It is hot", and q = " It is sunny"

SENTENCE	SYMBOLIC FORM
1.It is **not** hot.	$\sim p$
2.It is hot **and** sunny.	$p \wedge q$
3.It is hot **or** sunny.	$p \vee q$
4.It is **not** hot **but** sunny.	$\sim p \wedge q$
5.It is **neither** hot **nor** sunny.	$\sim p \wedge \sim q$

EXAMPLE

Let h = "Zia is healthy"
w = "Zia is wealthy"
s = "Zia is wise"

Translate the compound statements to symbolic form:

1) Zia is healthy and wealthy but not wise. $(h \wedge w) \wedge (\sim s)$
2) Zia is not wealthy but he is healthy and wise. $\sim w \wedge (h \wedge s)$
3) Zia is neither healthy, wealthy nor wise. $\sim h \wedge \sim w \wedge \sim s$

TRANSLATING FROM SYMBOLS TO ENGLISH:

Let m = "Ali is good in Mathematics"
c = "Ali is a Computer Science student"

Translate the following statement forms into plain English:

1) $\sim c$ Ali is **not** a Computer Science student
2) $c \vee m$ Ali is a Computer Science student **or** good in Maths.
3) $m \wedge \sim c$ Ali is good in Maths **but not** a Computer Science student

A convenient method for analyzing a compound statement is to make a truth table for it.

Truth Table

A **truth table** specifies the truth value of a compound proposition for all possible truth values of its constituent propositions.

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NEGATION (~):

If *p* is a statement variable, then negation of *p*, *"not p"*, is denoted as "~p"
It has opposite truth value from p i.e., if p is true, then ~ p is false; if p is false, then ~ p is true.

TRUTH TABLE FOR ~ p

p	$\sim p$
T	F
F	T

CONJUNCTION (∧):

If *p* and *q* are statements, then the conjunction of *p* and *q* is *"p and q"*, denoted as "$p \wedge q$".

Remarks

- $p \wedge q$ is true only when both p and q are true.
- If either p or q is false, or both are false, then $p \wedge q$ is false.

TRUTH TABLE FOR p ∧ q

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

DISJUNCTION (∨) or INCLUSIVE OR

If *p* & *q* are statements, then the disjunction of *p* and *q* is "*p or q*", denoted as "$p \vee q$".

Remarks:

- $p \vee q$ is true when at least one of p or q is true.
- $p \vee q$ is false only when both p and q are false.

TRUTH TABLE FOR p ∨ q

p	*q*	*p ∨ q*
T	T	T
T	F	T
F	T	T
F	F	F

Note it that in the table F is only in that row where both p and q have F and all other values are T. Thus for finding out the truth values for the disjunction of two statements we will only first search out where the both statements are false and write down the F in the corresponding row in the column of p ∨ q and in all other rows we will write T in the column of p ∨ q.

Remark:

Note that for Conjunction of two statements we find the T in both the statements, But in disjunction we find F in both the statements. In other words, we will fill T in the first row of conjunction and F in the last row of disjunction.

SUMMARY

1. What is a statement?
2. How a compound statement is formed.
3. Logical connectives (negation, conjunction, disjunction).
4. How to construct a truth table for a statement form.

Lecture No.2 Truth Tables

Truth Tables for:

1. $\sim p \wedge q$
2. $\sim p \wedge (q \vee \sim r)$
3. $(p \vee q) \wedge \sim (p \wedge q)$

Truth table for the statement form $\sim p \wedge q$

p	q	~p	~p ∧ q
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

Truth table for $\sim p \wedge (q \vee \sim r)$

p	q	r	~r	q ∨ ~r	~p	~p ∧ (q ∨ ~r)
T	T	T	F	T	F	F
T	T	F	T	T	F	F
T	F	T	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	T	F	T	T	T	T
F	F	T	F	F	T	F
F	F	F	T	T	T	T

Truth table for $(p \vee q) \wedge \sim (p \wedge q)$

p	q	p∨q	p∧q	~(p∧q)	(p∨q) ∧ ~(p∧q)
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

USAGE OF "OR" IN ENGLISH

In English language the word **OR** is sometimes used in an inclusive sense (p or q or both).

Example: I shall buy a pen or a book.

In the above statement, if you buy a pen or a book in both cases the statement is true and if you buy both pen and book, then statement is again true. Thus we say in the above statement we use or in inclusive sense.

The word **OR** is sometimes used in an exclusive sense (p or q but not both). As in the below statement

Example: Tomorrow at 9, I'll be in Lahore or Islamabad.

Now in above statement we are using **OR** in exclusive sense because if both the statements are true, then we have F for the statement.

While defining a disjunction the word **OR** is used in its inclusive sense. Therefore, the symbol $\vee$ means the "inclusive **OR**"

EXCLUSIVE OR:

When **OR** is used in its exclusive sense, The statement "p or q" means "p or q but not both" or "p or q and not p and q" which translates into symbols as $(p \vee q) \wedge \sim(p \wedge q)$

It is abbreviated as $p \oplus q$ or **p XOR q**

TRUTH TABLE FOR EXCLUSIVE OR:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

TRUTH TABLE FOR $(p\vee q) \wedge \sim(p \wedge q)$

p	q	$p\vee q$	$p\wedge q$	$\sim(p \wedge q)$	$(p\vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Note: Basically

$$p \oplus q \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$$
$$\equiv [p \wedge \sim q) \vee \sim p] \wedge [(p \wedge \sim q) \vee q]$$
$$\equiv (p \vee q) \wedge \sim (p \wedge q)$$
$$\equiv (p \vee q) \wedge (\sim p \vee \sim q)$$

LOGICAL EQUIVALENCE

If two logical expressions have the same logical values in the truth table, then we say that the two logical expressions are logically equivalent. In the following example, $\sim(\sim p)$ is logically equivalent p. So it is written as $\sim(\sim p) \equiv p$

Double Negative Property $\sim(\sim p) \equiv p$

p	~p	~(~p)
T	F	T
F	T	F

Example

Rewrite in a simpler form:

"It is not true that I am not happy."

Solution:

Let p = "I am happy"
then ~ p = "I am not happy"
and ~ (~ p) = "It is not true that I am not happy"
Since $\sim(\sim p) \equiv p$
Hence the given statement is equivalent to "I am happy"

Example

Show that $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent

Solution:

p	q	~p	~q	p∧q	~(p∧q)	~p ∧ ~q
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	T

Different truth values in row 2 and row 3

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DE MORGAN'S LAWS

1) The negation of an **AND** statement is logically equivalent to the **OR** statement in which each component is negated.

Symbolically $\sim(p \wedge q) \equiv \sim p \vee \sim q$

2) The negation of an **OR** statement is logically equivalent to the **AND** statement in which each component is negated.

Symbolically $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Truth Table of $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	~p	~q	p ∨ q	~(p ∨ q)	~p ∧ ~q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Same truth values

APPLICATION:

Give negations for each of the following statements:

a) The fan is slow **or** it is very hot.
b) Akram is unfit **and** Saleem is injured.

Solution:

a) The fan is **not** slow **and** it is **not** very hot.
b) Akram is **not** unfit **or** Saleem is **not** injured.

INEQUALITIES AND DEMORGAN'S LAWS:

Use DeMorgan's Laws to write the negation of

$-1 < x \le 4$ for some particular real number x

Here, $-1 < x \le 4$ means $x > -1$ **and** $x \le 4$

The negation of ($x > -1$ **and** $x \le 4$) is ($x \le -1$ **OR** $x > 4$).

We can explain it as follows:

Suppose p : $x > -1$

q : $x \le 4$

~p : $x \le -1$

~q : $x > 4$

The negation of $x > -1$ **AND** $x \le 4$

$\equiv \sim(p \wedge q)$

$\equiv \sim p \vee \sim q$ by DeMorgan's Law,

$\equiv x \leq -1$ OR $x > 4$

EXERCISE:

1. Show that $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
2. Are the statements $(p \wedge q) \vee r$ and $p \wedge (q \vee r)$ logically equivalent?

TAUTOLOGY:

A tautology is a statement form that is always true regardless of the truth values of the statement variables. A tautology is represented by the symbol "t".

EXAMPLE: The statement form $p \vee \sim p$ is tautology

p	~p	p ∨ ~p
T	F	T
F	T	T

$p \vee \sim p \equiv t$

CONTRADICTION:

A contradiction is a statement form that is always false regardless of the truth values of the statement variables. A contradiction is represented by the symbol "c".

So if we have to prove that a given statement form is **CONTRADICTION,** we will make the truth table for the statement form and if in the column of the given statement form all the entries are F, then we say that statement form is contradiction.

EXAMPLE:

The statement form $p \wedge \sim p$ is a contradiction.

p	~p	p ∧ ~p
T	F	F
F	T	F

Since in the last column in the truth table we have F in all the entries, so it is a contradiction i.e. $p \wedge \sim p \equiv c$

REMARKS:

- Most statements are neither tautologies nor contradictions.
- The negation of a tautology is a contradiction and vice versa.
- In common usage we sometimes say that two statement are contradictory. By this we mean that their conjunction is a contradiction: they cannot both be true.

LOGICAL EQUIVALENCE INVOLVING TAUTOLOGY

1. Show that $p \wedge t \equiv p$

p	t	$p \wedge t$
T	T	T
F	T	F

Since in the above table the entries in the first and last columns are identical so we have the corresponding statement forms are Logically equivalent that is

$p \wedge t \equiv p$

LOGICAL EQUIVALENCE INVOLVING CONTRADICTION

Show that $p \wedge c \equiv c$

p	c	$p \wedge c$
T	F	F
F	F	F

There are same truth values in the indicated columns, so $p \wedge c \equiv c$

EXERCISE:

Use truth table to show that $(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$ is a tautology.

SOLUTION:

Since we have to show that the given statement form is Tautology, so the column of the above proposition in the truth table will have all entries as T. As clear from the table below

p	q	$p \wedge q$	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \vee (p \wedge \sim q)$	$(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$
T	T	T	F	F	F	F	T
T	F	F	F	T	T	T	T
F	T	F	T	F	F	T	T
F	F	F	T	T	F	T	T

Hence $(p \wedge q) \vee (\sim p \vee (p \wedge \sim q)) \equiv t$

EXERCISE:

Use truth table to show that $(p \wedge \sim q) \wedge (\sim p \vee q)$ is a contradiction.

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SOLUTION:

Since we have to show that the given statement form is Contradiction, so its column in the truth table will have all entries as F. As clear from the table below.

p	q	~q	p ∧ ~q	~p	~p ∨ q	(p ∧ ~q) ∧ (~p ∨ q)
T	T	F	F	F	T	F
T	F	T	T	F	F	F
F	T	F	F	T	T	F
F	F	T	F	T	T	F

LAWS OF LOGIC

1) **Commutative Laws**

$$p \wedge q \equiv q \wedge p$$
$$p \vee q \equiv q \vee p$$

2) **Associative Laws**

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$
$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

3) **Distributive Laws**

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

4) **Identity Laws**

$$p \wedge t \equiv p$$
$$p \vee c \equiv p$$

5) **Negation Laws**

$$p \vee \sim p \equiv t$$
$$p \wedge \sim p \equiv c$$

6) **Double Negation Law**

$$\sim(\sim p) \equiv p$$

7) **Idempotent Laws**

$$p \wedge p \equiv p$$
$$p \vee p \equiv p$$

8) **DeMorgan's Laws**

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$
$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

9) **Universal Bound Laws**

$$p \vee t \equiv t$$
$$p \wedge c \equiv c$$

10) Absorption Laws

$p \vee (p \wedge q) \equiv p$

$p \wedge (p \vee q) \equiv p$

11) Negation of t and c

$\sim t \equiv c$

$\sim c \equiv t$

Lecture No.3 Laws of Logic

APPLYING LAWS OF LOGIC

Using law of logic, simplify the statement form

$p \vee [\sim(\sim p \wedge q)]$

Solution:

$p \vee [\sim(\sim p \wedge q)]$	$\equiv p \vee [\sim(\sim p) \vee (\sim q)]$	DeMorgan's Law
	$\equiv p \vee [p \vee (\sim q)]$	Double Negative Law: $\sim(\sim p) \equiv p$
	$\equiv [p \vee p] \vee (\sim q)$	Associative Law for $\vee$
	$\equiv p \vee (\sim q)$	Idempotent Law: $p \vee p \equiv p$

That is the simplified statement form.

Example: Using Laws of Logic, verify the logical equivalence

$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$

Solution:

$\sim(\sim p \wedge q) \wedge (p \vee q)$	$\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q)$	DeMorgan's Law
	$\equiv (p \vee \sim q) \wedge (p \vee q)$	Double Negative Law
	$\equiv p \vee (\sim q \wedge q)$	Distributive Law
	$\equiv p \vee c$	Negation Law
	$\equiv p$	Identity Law

SIMPLIFYING A STATEMENT:

"You will get an A if you are hardworking and the sun shines, or you are hardworking and it rains." Rephrase the condition more simply.

Solution:

Let p = "You are hardworking'
q = "The sun shines"
r = "It rains" .

The condition is $(p \wedge q) \vee (p \wedge r)$

Using distributive law in reverse,

$$(p \wedge q) \vee (p \wedge r) \equiv p \wedge (q \vee r)$$

Putting $p \wedge (q \vee r)$ back into English, we can rephrase the given sentence as

"You will get an A if you are hardworking and the sun shines or it rains.

EXERCISE:

Use Logical Equivalence to rewrite each of the following sentences more simply.

1.It is not true that I am tired and you are smart.
{I am not tired or you are not smart.}

2.It is not true that I am tired or you are smart.
{I am not tired and you are not smart.}

3.I forgot my pen or my bag and I forgot my pen or my glasses.
{I forgot my pen or I forgot my bag and glasses.

4.It is raining and I have forgotten my umbrella, or it is raining and I have forgotten my hat.
{It is raining **and** I have forgotten my umbrella **or** my hat.}

CONDITIONAL STATEMENTS:

Introduction

Consider the statement:

"If you earn an A in Math, then I'll buy you a computer."

This statement is made up of two simpler statements:

p: "You earn an A in Math"
q: "I will buy you a computer."

The original statement is then saying :

if p is true, then q is true, or, more simply, **if** p, **then** q.

We can also phrase this as p **implies** q. It is denoted by **$p \to q$.**

CONDITIONAL STATEMENTS OR IMPLICATIONS:

If p and q are statement variables, the conditional of q by p is "If p then q" or "p implies q" and is denoted $p \to q$.

$p \to q$ is false when p is true and q is false; otherwise it is true.
The arrow "$\to$" is the **conditional** operator.
In $p \to q$, the statement **p** is called **the hypothesis (or antecedent)** and q is called the **conclusion** (**or consequent**).

TRUTH TABLE:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

PRACTICE WITH CONDITIONAL STATEMENTS:

Determine the truth value of each of the following conditional statements:

1. "If 1 = 1, then 3 = 3." **TRUE**
2. "If 1 = 1, then 2 = 3." **FALSE**
3. "If 1 = 0, then 3 = 3." **TRUE**
4. "If 1 = 2, then 2 = 3." **TRUE**
5. "If 1 = 1,then 1 = 2 and 2 = 3." **FALSE**
6. "If 1 = 3 or 1 = 2 then 3 = 3." **TRUE**

ALTERNATIVE WAYS OF EXPRESSING IMPLICATIONS:

The implication p → q could be expressed in many alternative ways as:

- "if p then q"
- "p implies q"
- "if p, q"
- "p only if q"
- "p is sufficient for q"
- "not p unless q"
- "q follows from p"
- "q if p"
- "q whenever p"
- "q is necessary for p"

EXERCISE:

Write the following statements in the form "if p, then q" in English.

a)Your guarantee is good only if you bought your CD less than 90 days ago.

If your guarantee is good, then you must have bought your CD player less than 90 days ago.

b)To get tenure as a professor, it is sufficient to be world-famous.

If you are world-famous, then you will get tenure as a professor.

c)That you get the job implies that you have the best credentials.

If you get the job, then you have the best credentials.

d)It is necessary to walk 8 miles to get to the top of the Peak.

If you get to the top of the peak, then you must have walked 8 miles.

TRANSLATING ENGLISH SENTENCES TO SYMBOLS:

Let p and q be propositions:

p = "you get an A on the final exam"
q = "you do every exercise in this book"
r = "you get an A in this class"

Write the following propositions using p, q, and r and logical connectives.

1.To get an A in this class it is necessary for you to get an A on the final.

SOLUTION $p \rightarrow r$

2.You do every exercise in this book; You get an A on the final, implies, you get an A in the class.

SOLUTION $p \wedge q \rightarrow r$

3. Getting an A on the final and doing every exercise in this book is sufficient For getting an A in this class.

SOLUTION $p \wedge q \rightarrow r$

TRANSLATING SYMBOLIC PROPOSITIONS TO ENGLISH:

Let p, q, and r be the propositions:

p = "you have the flu"
q = "you miss the final exam"
r = "you pass the course"

Express the following propositions as an English sentence.

1. $p \rightarrow q$

If you have flu, then you will miss the final exam.

2. $\sim q \rightarrow r$

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If you don't miss the final exam, you will pass the course.

3. ~p ∧ ~q→ r

If you neither have flu nor miss the final exam, then you will pass the course.

HIERARCHY OF OPERATIONS
FOR LOGICAL CONNECTIVES

- ~ (negation)
- ∧ (conjunction), ∨ (disjunction)
- → (conditional)

Example: Construct a truth table for the statement form **p ∨ ~ q → ~ p**

p	q	~q	~p	p ∨ ~q	p ∨ ~ q → ~ p
T	T	F	F	T	F
T	F	T	F	T	F
F	T	F	T	F	T
F	F	T	T	T	T

Example: Construct a truth table for the statement form **(p →q)∧(~ p → r)**

p	q	r	p→q	~p	~p→r	(p→q)∧(~ p →r)
T	T	T	T	F	T	T
T	T	F	T	F	T	T
T	F	T	F	F	T	F
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	F

LOGICAL EQUIVALENCE INVOLVING IMPLICATION

Use truth table to show **p→q ≡ ~q → ~p**

p	q	~q	~p	p→q	~q → ~p
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

same truth values

Hence the given two expressions are equivalent.

IMPLICATION LAW

p→q ≡ ~p∨q

p	q	p→q	~p	~p∨q
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

same truth values

NEGATION OF A CONDITIONAL STATEMENT:

Since $p \to q \equiv \sim p \vee q$

So $\sim(p \to q) \equiv \sim(\sim p \vee q)$

$\equiv \sim(\sim p) \wedge (\sim q)$ by De Morgan's law

$\equiv p \wedge \sim q$ by the Double Negative law

Thus the negation of "**if p then q**" is logically equivalent to "**p and not q**".
Accordingly, the negation of an **if-then** statement does not start with the word **if**.

EXAMPLES

Write negations of each of the following statements:

1.If Ali lives in Pakistan then he lives in Lahore.
2.If my car is in the repair shop, then I cannot get to class.
3.If x is prime then x is odd or x is 2.
4.If n is divisible by 6, then n is divisible by 2 and n is divisible by 3.

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SOLUTIONS:

1. Ali lives in Pakistan and he does not live in Lahore.
2. My car is in the repair shop and I can get to class.
3. x is prime but x is not odd and x is not 2.
4. n is divisible by 6 but n is not divisible by 2 or by 3.

INVERSE OF A CONDITIONAL STATEMENT:

The inverse of the conditional statement **p → q** is **~p → ~q**

A conditional and its inverse are not equivalent as could be seen from the truth table.

p	q	p→q	~p	~q	~p →~q
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

different truth values in rows 2 and 3

WRITING INVERSE:

1. *If today is Friday, then 2 + 3 = 5.*
   If today is not Friday, then 2 + 3 ≠ 5.

2. *If it snows today, I will ski tomorrow.*
   If it does not snow today I will not ski tomorrow.

3. *If P is a square, then P is a rectangle.*
   If P is not a square then P is not a rectangle.

4. *If my car is in the repair shop, then I cannot get to class.*
   If my car is not in the repair shop, then I shall get to the class.

CONVERSE OF A CONDITIONAL STATEMENT:

The converse of the conditional statement **p → q** is **q →p.**

A conditional and its converse are not equivalent. i.e., → is not a commutative operator.

p	q	p→q	q→p
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

not the same

WRITING CONVERSE:

1.If today is Friday, then 2 + 3 = 5.
If 2 + 3 = 5, then today is Friday.

2.If it snows today, I will ski tomorrow.
I will ski tomorrow only if it snows today.

3. If P is a square, then P is a rectangle.
If P is a rectangle then P is a square.

4. If my car is in the repair shop, then I cannot get to class.
If I cannot get to the class, then my car is in the repair shop.

CONTRAPOSITIVE OF A CONDITIONAL STATEMENT:

The contra-positive of the conditional statement $p \to q$ is $\sim q \to \sim p$
A conditional and its contra-positive are equivalent.

Symbolically $p \to q \equiv \sim q \to \sim p$

1.If today is Friday, then 2 + 3 = 5.
If $2 + 3 \neq 5$, then today is not Friday.

2.If it snows today, I will ski tomorrow.
I will not ski tomorrow only if it does not snow today.

3. If P is a square, then P is a rectangle.
If P is not a rectangle then P is not a square.

4. If my car is in the repair shop, then I cannot get to class.
If I can get to the class, then my car is not in the repair shop.

EXERCISE:

1. Show that $p \to q \equiv \sim q \to \sim p$ (Use the truth table.)
2. Show that $q \to p \equiv \sim p \to \sim q$ (Use the truth table.)

Lecture No.4 Biconditional

BICONDITIONAL

If p and q are statement variables, the biconditional of p and q is **"p if and only if q"**.
It is denoted **p↔q**. "*if and only if*" is abbreviated as **iff**.
The double headed arrow " ↔" is the **biconditional operator.**

TRUTH TABLE FOR p↔q.

p	q	p ↔q
T	T	T
T	F	F
F	T	F
F	F	T

Remark:

- p ↔ q is true only when p and q both are true or both are false.
- p ↔ q is false when either p or q is false.

EXAMPLES:
Identify which of the following are True or false?

1. **"1+1 = 3** if and only if **earth is flat"**
   TRUE
2. **"Sky is blue iff 1 = 0"**
   FALSE
3. **"Milk is white iff birds lay eggs"**
   TRUE
4. **"33 is divisible by** 4 if and only if **horse has four legs"**
   FALSE
5. **"$x > 5$ iff $x^2 > 25$"**
   FALSE

REPHRASING BICONDITIONAL:

p↔q is also expressed as:

- "p is necessary and sufficient for q"
- "If p then q, and conversely"
- "p is equivalent to q"

Example: Show that $p \leftrightarrow q \equiv (p\to q) \wedge (q\to p)$

p	q	p↔q	p→q	q→p	(p→q)∧(q→p)
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

same truth values

EXERCISE:

Rephrase the following propositions in the form "p if and only if q" in English.

1. If it is hot outside, you buy an ice cream cone, and if you buy an ice cream cone, it is hot outside.

Sol You buy an ice cream cone if and only if it is hot outside.

2. For you to win the contest it is necessary and sufficient that you have the only winning ticket.

Sol You win the contest if and only if you hold the only winning ticket.

3. If you read the news paper every day, you will be informed and conversely.

Sol You will be informed if and only if you read the news paper every day.

4. It rains if it is a weekend day, and it is a weekend day if it rains.

Sol It rains if and only if it is a weekend day.

5. The train runs late on exactly those days when I take it.

Sol The train runs late if and only if it is a day I take the train.

6. This number is divisible by 6 precisely when it is divisible by both 2 and 3.

Sol This number is divisible by 6 if and only if it is divisible by both 2 and 3.

TRUTH TABLE FOR $(p\to q) \leftrightarrow (\sim q\to \sim p)$

p	q	p→q	~q	~p	~q→~p	(p→q)↔(~q→~p)
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

TRUTH TABLE FOR (p↔q) ↔ (r ↔ q)

p	q	r	p ↔ q	r ↔ q	(p ↔q)↔(r↔ q)
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	F	T
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	F	T
F	F	T	T	F	F
F	F	F	T	T	T

TRUTH TABLE FOR p ∧ ~r ↔ q ∨ r

Here $p \wedge \sim r \leftrightarrow q \vee r$ means $(p \wedge (\sim r)) \leftrightarrow (q \vee r)$

p	q	r	~r	p∧~r	q∨r	p ∧ ~r ↔q∨r
T	T	T	F	F	T	F
T	T	F	T	T	T	T
T	F	T	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	T	F	T	F	T	F
F	F	T	F	F	T	F
F	F	F	T	F	F	T

LOGICAL EQUIVALENCE INVOLVING BICONDITIONAL

Example: Show that ~p ↔ q and p ↔ ~q are logically equivalent.

p	q	~p	~q	~p↔q	p ↔ ~q
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

same truth values

Hence ~p ↔ q ≡ p ↔ ~q

EXERCISE:

Show that ~(p⊕q) and p↔q are logically equivalent.

p	q	p⊕q	~(p⊕q)	p↔q
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

same truth values

Hence ~(p⊕q) ≡ p↔q

LAWS OF LOGIC:

1.Commutative Law: p ↔ q ≡ q ↔ p

2.Implication Laws: p → q ≡ ~ p ∨ q
≡ ~ (p ∧ ~ q)

3.Exportation Law: (p ∧ q)→r ≡ p →(q →r)

4.Equivalence: p ↔ q ≡ (p →q)∧(q →p)

5.Reductio ad absurdum p →q ≡ (p ∧ ~q) →c

APPLICATION:

Example: Rewrite the statement forms without using the symbols → or ↔

1. p ∧ ~ q→ r
2. (p→ r) ↔ (q → r)

Solution:

1. $p \wedge \sim q \to r \equiv (p \wedge \sim q) \to r$ Order of operations
   $\equiv \sim (p \wedge \sim q) \vee r$ Implication law

2. $(p \to r) \leftrightarrow (q \to r) \equiv (\sim p \vee r) \leftrightarrow (\sim q \vee r)$ Implication law
   $\equiv [(\sim p \vee r) \to (\sim q \vee r)] \wedge [(\sim q \vee r) \to (\sim p \vee r)]$
   Equivalence of biconditional
   $\equiv [\sim(\sim p \vee r) \vee (\sim q \vee r)] \wedge [\sim(\sim q \vee r) \vee (\sim p \vee r)]$
   Implication law

Example: Rewrite the statement form $\sim p \vee q \to r \vee \sim q$ to a logically equivalent form that uses only ~ and ∧.

Solution:

STATEMENT	REASON
$\sim p \vee q \to r \vee \sim q$	Given statement form
$\equiv (\sim p \vee q) \to (r \vee \sim q)$	Order of operations
$\equiv \sim[(\sim p \vee q) \wedge \sim (r \vee \sim q)]$	Implication law $p \to q \equiv \sim(p \wedge \sim q)$
$\equiv \sim[\sim(p \wedge \sim q) \wedge (\sim r \wedge q)]$	De Morgan's law

Example: Show that $\sim(p \to q) \to p$ is a tautology without using truth tables.

Solution:

STATEMENT	REASON
$\sim(p \to q) \to p$	Given statement form
$\equiv \sim[\sim(p \wedge \sim q)] \to p$	Implication law $p \to q \equiv \sim(p \wedge \sim q)$
$\equiv (p \wedge \sim q) \to p$	Double negation law
$\equiv \sim(p \wedge \sim q) \vee p$	Implication law $p \to q \equiv \sim p \vee q$
$\equiv (\sim p \vee q) \vee p$	De Morgan's law
$\equiv (q \vee \sim p) \vee p$	Commutative law of ∨
$\equiv q \vee (\sim p \vee p)$	Associative law of ∨
$\equiv q \vee t$	Negation law
$\equiv t$	Universal bound law

EXERCISE:

Suppose that p and q are statements so that $p \to q$ is false. Find the truth values of each of the following:

1. $\sim p \to q$
2. $p \vee q$
3. $q \leftrightarrow p$

SOLUTION

Hint: ($p \to q$ is false when p is true and q is false.)

1. TRUE
2. TRUE
3. FALSE

Lecture No.5 Argument

Before we discuss in detail about the argument, we first consider the following argument:

An interesting teacher keeps me awake. I stay awake in Discrete Mathematics class. Therefore, my Discrete Mathematics teacher is interesting.

Is the above argument valid?

ARGUMENT:

An **argument** is a list of statements called **premises** (or **assumptions** or **hypotheses**) followed by a statement called the **conclusion.**

P_1 Premise

P_2 Premise

P_3 Premise

. . . .

P_n Premise

$\therefore C$ Conclusion

NOTE: The symbol $\therefore$ read "therefore" is normally placed just before the conclusion.

VALID AND INVALID ARGUMENT:

An argument is **valid** if the conclusion is true when all the premises are true.
Alternatively, an argument is valid if conjunction of its premises imply conclusion.
That is $(P_1 \wedge P_2 \wedge P_3 \wedge \ldots \wedge P_n) \to C$ is a tautology.
An argument is **invalid** if the conclusion is false when all the premises are true.
Alternatively, an argument is invalid if conjunction of its premises does not imply conclusion.

Critical Rows: The critical rows are those rows where the premises have truth value T.

EXAMPLE: Show that the following argument form is valid:

$p \to q$

p

$\therefore q$

SOLUTION

premises → ($p \to q$, p) conclusion → (q)

p	q	p→q	p	q	
T	T	T	T	T	← critical row
T	F	F	T	F	
F	T	T	F	T	
F	F	T	F	F	

Since the conclusion q is true when the premises p→q and p are True. Therefore, it is a valid argument.

EXAMPLE Show that the following argument form is invalid:

p→q
q
∴ p

SOLUTION

premises: p→q, q; conclusion: p

p	q	p→q	q	p
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	F

critical row

In the second critical row, the conclusion is false when the premises p→q and q are true. Therefore, the argument is invalid.

EXERCISE:

Use truth table to determine the argument form

$p \vee q$
$p \rightarrow \sim q$
$p \rightarrow r$
∴ r

is valid or invalid.

premises: p∨q, p→~q, p→r; conclusion: r

p	q	r	p∨q	p→~q	p→r	r
T	T	T	T	F	T	T
T	T	F	T	F	F	F
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	F
F	F	T	F	T	T	T
F	F	F	F	T	T	F

critical rows

In the third critical row, the conclusion is false when all the premises are true. Therefore, the argument is invalid.

The argument form is invalid

WORD PROBLEM

If Tariq is not on team A, then Hameed is on team B.
If Hameed is not on team B, then Tariq is on team A.
∴ Tariq is not on team A or Hameed is not on team B.

SOLUTION

Let

t = Tariq is on team A
h = Hameed is on team B

Then the argument is

$\sim t \to h$
$\sim h \to t$
$\therefore \sim t \vee \sim h$

t	h	$\sim t \to h$	$\sim h \to t$	$\sim t \vee \sim h$
T	T	T	T	F
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

Argument is invalid because there are three critical rows.

(Remember that the critical rows are those rows where the premises have truth value T) and in the first critical row conclusion has truth value F.

(Also remember that we say an argument is valid if in all critical rows conclusion has truth value T)

EXERCISE

If at least one of these two numbers is divisible by 6, then the product of these two numbers is divisible by 6.
Neither of these two numbers is divisible by 6.
∴ The product of these two numbers is not divisible by 6.

SOLUTION

Let d = at least one of these two numbers is divisible by 6.
p = product of these two numbers is divisible by 6.

Then the argument become in these symbols

$d \to p$
$\sim d$
$\therefore \sim p$

We will made the truth table for premises and conclusion as given below

d	p	d → p	~d	~p
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

In the first critical row, the conclusion is false when the premises are true. Therefore, the argument is invalid.

EXERCISE

If I got an Eid bonus, I'll buy a stereo.
If I sell my motorcycle, I'll buy a stereo.

∴ If I get an Eid bonus or I sell my motorcycle, then I'll buy a stereo.

SOLUTION:

Let

e = I got an Eid bonus
s = I'll buy a stereo
m = I sell my motorcycle

The argument is

$e \to s$
$m \to s$
$\therefore e \vee m \to s$

e	s	m	e →s	m →s	e∨m	e∨m →s
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	F	T	F
T	F	F	F	T	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	T
F	F	T	T	F	T	F
F	F	F	T	T	F	T

The argument is valid because in the five critical rows, the conclusion is true.

EXERCISE

An interesting teacher keeps me awake. I stay awake in Discrete Mathematics class. Therefore, my Discrete Mathematics teacher is interesting.

Solution:

t = My teacher is interesting
a = I stay awake
m = I am in Discrete Mathematics class

The argument to be tested is

$t \to a,$
$a \wedge m$
Therefore $m \wedge t$

t	a	m	t → a	a ∧ m	m ∧ t
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	F	T
T	F	F	F	F	F
F	T	T	T	T	F
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	T	F	F

In the second critical row, the conclusion is false when the premises are true. Therefore, the argument is invalid.